A PIECE-WISE LINEAR MODEL FOR APPROXIMATING LMP-LOAD CURVE BASED ON CRITICAL LOAD LEVELS

Rui Bo
Midwest ISO, St.Paul, USA, rui.bo@ieee.org
Fangxing Li
The University of Tennessee, Knoxville, USA, fli6@utk.edu

***Abstract* —In deregulated power markets, it is of high interest to have a mathematical representation of the LMP-Load curve for the purpose of various analytical studies such as probabilistic LMP analysis. However, LMP versus Load curve may present various linear and nonlinear patterns, and appears challenging to develop a simple and unified form of mathematical model to capture those patterns observed in the LMP-Load curve. Studies have shown that the LMP-Load curve presents change of patterns at Critical Load Levels (CLLs) where step changes of LMP occur. By utilizing the concept of CLLs, this paper proposes a method to divide a LMP-Load curve into segments, and employ linear curve-fitting to obtain a piece-wise linear model to represent the actual LMP-Load curve. Through accurately identified break points of the curve, the proposed method can very closely approximate the actual curve, which traditional curve fitting methods would fail to do. The method has been tested on a revised IEEE 118-bus system. The resulting piece-wise linear model has been verified to be an exceptionally accurate model to represent the actual LMP-Load curve. Various patterns of the curve have been captured and closely represented by the piece-wise linear approximation model.**

Key Words: Locational Marginal Price (LMP), Optimal Power Flow (OPF), ACOPF, DCOPF, nonlinear, Probabilistic LMP Forecasting, Critical Load Level (CLL)

I. INTRODUCTION

In US power markets, locational marginal price (LMP) serves as price signals for short-term electricity trading and long-term investment for market participants [1-4]. As load is the one of the primary drivers of LMP variations, LMP versus Load curve is a fundamental piece for various studies such as LMP trend analysis, LMP prediction, load forecasting impact on LMP, etc. Existing studies have shown that, in deregulated power markets, LMP-Load curve obtained from lossless DCOPF model is a staircase curve with respect to changes of total system load [5, 6]. Bus LMP is a constant value at each stair and jumps to a different value when system load reaches a new stair. The curve, however, can become very nonlinear in ACOPF framework where power losses, reactive power, and voltages are taken into account. The resulting LMP-Load curve is no longer a piece-wise constant curve, instead, it often exhibits high nonlinearity and appears hard to find a mathematical representation. On the other hand, some studies require a mathematical form of LMP-Load curve in order to perform analytical studies. For example, a Probabilistic LMP Forecasting methodology is proposed to study and quantify the impact of load uncertainty on LMP which requires a mathematical model for the LMP-Load curve [7]. As such, a mathematical representation of the LMP-Load curve is of high interest especially when it has a simple form and can represent the LMP-Load curve very closely.

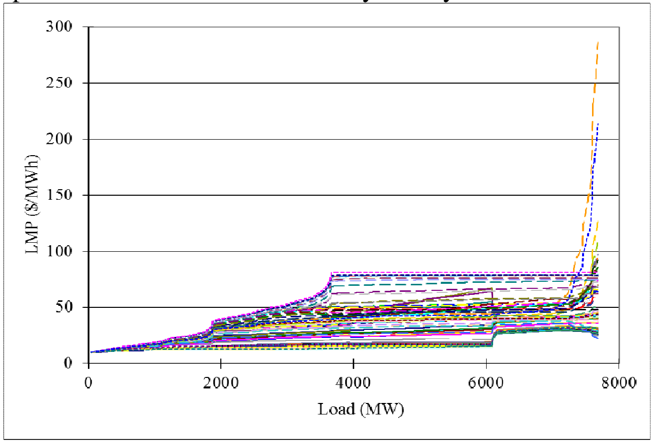

Fig. 1. LMP-Load curves for each bus of the IEEE 118-bus system.

It seems very challenging to represent LMP-Load curve by simple mathematical forms when the curve appears to be highly nonlinear around some load levels. For instance, Figure 1 shows the LMP versus System Load curve for each bus of the IEEE 118-bus test system. The curves of some buses demonstrate polynomial pattern when load is around 1800MW and 3500MW, and exhibit even more complicated patterns when load is greater than 7000MW.

One might consider to apply curve fitting method or curve smoothing method [8] to construct a mathematical formula to represent the actual LMP-Load curve. However, it is difficult for these methods to preserve the drastic curve patterns of the actual curve such as step changes because these methods tend to create a curve to smooth out given data points. In addition, these purely data-driven methods rely heavily on the data points and therefore lack the ability of self-identifying locations where curve pattern changes significantly. To preserve all curve patterns, the key is to find out the break points where the LMP-Load curve changes pattern.

Fortunately existing works on LMP trend analysis have offered a unique clue to help solve the problem. LMP may change significantly, and sometimes jump, at some load levels. These load levels have been called Critical Load Levels (CLLs) which occur when there is a shift of marginal units or change of branch congestion status [5]. From mathematical perspective, a change in binding constraint set and non-binding constraint set occurs at a CLL.

Based on the CLL concept, in this paper, a piece-wise linear model approach is proposed to approximate LMP-Load curve. First, ACOPF simulations are performed on the

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studied load level range with sufficiently small simulation step in order to get the actual LMP-Load curve. Then, CLLs are identified by employing binary search algorithm on ACOPF solutions, specifically, the marginal unit set and congested branch set. Next, perform linear curve-fitting between every two identified CLLs. Last, a piece-wise linear curve is obtained with CLLs being the break points. The resulting model can very closely represent the actual curve due to its ability of identifying the break points where the curve changes its pattern.

This paper is organized as follows. First, the concept of CLL, as well as the approaches for locating CLLs, is reviewed in section II. A linear curve-fitting based approach is presented in section III and resulting piece-wise linear model is subsequently obtained. The proposed approach is illustrated on a revised IEEE 118-bus test system in section IV. Section V concludes the paper.

II. CRITICAL LOAD LEVEL AND ITS IDENTIFICATION

A. Critical Load Levels

When system load varies slightly around a given operating point, the same set of marginal units will pick up the load variation, and the congested branches will remain congested. Mathematically, the set of binding constraints and set of unbinding constraints remain intact. However, when system load varies significant enough to a new level where a shift on marginal units or new congestion happens, the binding constraint set and unbinding constraint set will change. This load level is called Critical Load Level (CLL).

The load level is ***critical*** because across this point system statuses may change significantly from their past trajectory. For instance, when the generation upper bound of a marginal unit is reached at a CLL, the marginal unit will likely become a non-marginal unit and another more expensive generator will become marginal. Thus the generation dispatch pattern will change. Likewise, at a CLL, LMP jump may occur due to shift of marginal units or new congestion.

For a given operating point, the highest CLL to its left is called the Previous CLL, and the lowest CLL to its right is called the Next CLL. In other words, the Previous CLL and Next CLL are the two closest CLLs in distance from the given point.

B. Approaches for Locating CLLs

Based on the definition of CLL, a criterion for identifying CLL can be derived as follows:

A system load level is a Critical Load Level where the binding constraint set or non-binding constraint set changes when system load varies right across this load level.

This criterion has been utilized to develop algorithms for locating CLLs. Ref. [5] applies parametric analysis on the linear programming model for a lossless DCOPF model, and proposes a simplex-like method to efficiently and accurately solve for the Previous CLL and Next CLL around any given operating point. Ref. [14] presents a variable substitution method to approximate the generation dispatch using quadratic polynomials for with-loss DCOPF model. Ref. [9] proposes a quadratic interpolation method to estimate CLL for ACOPF framework where ACOPF solutions at three selected load levels are used to get the trajectory of system statuses such as generation dispatch and line flows, which subsequently are used for CLL estimation. CLL estimated by this method may be slight off from the actual CLL simply because system statuses are not perfectly quadratic functions of system load.

In this paper, a binary search algorithm is used for locating CLLs. Binary search algorithm is a commonly used algorithm in locating position of items in a sorted array [15]. On average, it takes $log_2(N) - 1$ times of iterations before locating the item, where N is the total number of items. When it is applied to seeking CLL, the number of items depends on the length of the search interval and the precision requirement. For instance, if a CLL is to be located in a 10MW interval with 0.001MW precision, the total number of items will be 10001 (=10/0.001+1). So, the algorithm may be less efficient for extremely high precision requirement. Nevertheless, algorithm efficiency is not the focus of this paper and binary search algorithm is capable of getting CLL to any desired precision, therefore it is utilized in this paper to accurately pinpoint CLLs.

For a given operating point with load level D_{init}, the process of seeking the Next CLL is as follows:

1) Assign D_{init} to D_{lb};
2) Solve ACOPF at D_{lb};
3) Give an arbitrary load level D_{ub} greater than D_{lb}. D_{ub} is normally chosen to be large enough so that the Next CLL shall fall in the interval $[D_{lb}, D_{ub}]$;
4) Assign $(D_{lb} + D_{ub})/2$ to D_{mid}. Solve ACOPF at D_{mid};
5) If the binding constraint set and unbinding constraint set at D_{mid} are exactly the same as those at D_{lb}, assign D_{mid} to D_{lb}; Otherwise, assign D_{mid} to D_{ub};
6) If D_{ub} is sufficiently close to D_{lb} (e.g., 0.001 MW), done; otherwise, go to step 4.

The above process can be easily adapted for locating the Previous CLL. It should be noted that a second criterion is used implicitly in the above process. That is, when two load levels respectively reside in two different segments bounded by CLLs, the binding and non-binding constraint sets associated with the two load levels are different.

III. PIECEWISE LINEAR MODEL FOR LMP-LOAD CURVE

Once the CLLs are identified, they can be utilized to divide LMP-Load curve into segments, and the curve in each segment can be approximated by a linear polynomial curve obtained from linear curve-fitting.

A. Piece-wise linear curve-fitting on LMP-Load data points

A linear polynomial curve-fitting is employed to approximate the actual LMP-Load curve on the basis of segment, and the obtained coefficients are used to establish the mathematical model for the LMP versus load curve.

The process of linear curve-fitting for any segment demarcated by two immediately adjacent CLLs, e.g., CLL_s and CLL_{s+1}, is described as follows and illustrated in Figure 2.

1) Pick M load levels, $\{D_j\}_{j=1}^{M}$, in open interval (CLL_s, CLL_{s+1}). The load levels can be randomly chosen or evenly distributed. Make sure $M \geq 2$.
2) Solve ACOPF on the selected load levels and get the LMPs. The Load-LMP data points are denoted by $\{(D_j, LMP_j)\}_{j=1}^{M}$
3) Perform linear polynomial curve-fitting on data points $\{(D_j, LMP_j)\}_{j=1}^{M}$, and obtain the coefficients of the linear polynomial curve.

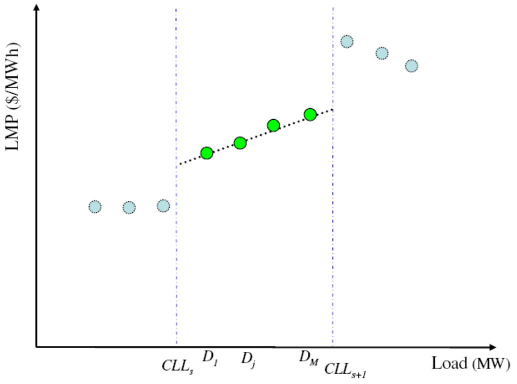


Fig. 2. Illustration of piece-wise linear curve-fitting on LMP versus Load data points.

In Figure 2, the data points at the selected load levels are shown in green circles and the linear polynomial is represented by a dotted line in black. Some data points outside interval [CLL_s, CLL_{s+1}] are drawn in light blue circles simply to illustrate the possible step changes at CLLs and different LMP pattern with respect to system load variation in different LMP segments.

It should be noted that the process of piece-wise linear curve-fitting should be combined with the CLL locating process where ACOPF runs have already been performed and thus the calculated LMPs can be directly used by the linear curve-fitting process.

Given the observation from Figure 1, one would not expect linear curve-fitting can suffice as a good approximation model for the actual LMP-Load curve since nonlinearity appears in many sections of the curve. Higher order polynomials or other more sophisticated formulas seem to be necessary in getting a reasonably accurate approximation model. However, as verified in section IV, piece-wise linear polynomial model can achieve an amazingly high accuracy in approximating the actual curve, with the knowledge of CLLs and its locations.

B. Mathematical Representation of LMP versus Load Curve

With the linear polynomial coefficients for each segment calculated from linear curve-fitting, the piece-wise linear representation of the LMP versus Load curve is readily available.

Figure 3 shows an illustrative graph of the LMP versus load curve in the ACOPF framework. The load axis is divided into n-1 segments by a sequence of critical load levels (CLLs), $\{D_i\}_{i=1}^{n}$. Here, D_1 represents the no-load case (i.e., $D_1=0$), and D_n represents the maximum load that the system can supply due to the limits of total generation resources and transmission capabilities. The LMP in each load segment i is depicted by a straight (linear) line, with slope a_i and intercept b_i.

The LMP-Load curve is extended to include two extra segments in Figure 3 [7]. One is for the load from D_0 to D_1, where D_0 denotes a negative infinite load, and the associated price is zero. The second additional segment is defined as the load range from D_n to D_{n+1}, where D_{n+1} represents a positive infinite load. In this segment, the price is set as the Value of the Lost Load (VOLL) to reflect demand response to load shedding. Although the VOLL varies with customer groups and load interruption time and duration, it is a common practice to produce an aggregated VOLL to represent the average loss for an area. Therefore, the VOLL is assumed to be a constant value for simplicity in this work. The two extra segments are added simply for mathematical completeness.

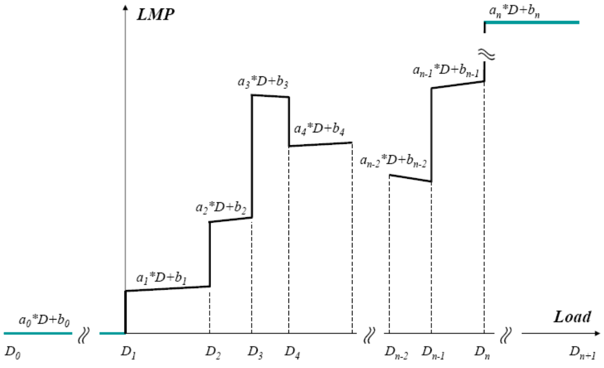


Fig. 3. Piecewise linear representation for an extended LMP-Load curve.

The LMP versus load curve can be formulated as

$$y(D) = \begin{cases} a_0 \times D + b_0, & D_0 < D \leq D_1 \\ a_1 \times D + b_1, & D_1 < D \leq D_2 \\ \vdots & \\ a_{n-1} \times D + b_{n-1}, & D_{n-1} < D \leq D_n \\ a_n \times D + b_n, & D_n < D \leq D_{n+1} \end{cases}$$

where

$a_0 = b_0 = a_n = 0$; $b_n = VOLL$

$D_0 = -\infty$; $D_1 = 0$; $D_{n+1} = \infty$.

The compact representation is given as follows

$$y(D) = \{a_i \times D + b_i \mid i \in \{0,1,\cdots,n\}, D_i < D \le D_{i+1}\}$$

The actual LMP sensitivity at load levels other than CLLs ($\{D_i\}_{i=1}^{n-1}$) may be calculated through perturbation method [10].

This piecewise linear model will greatly facilitate analytical studies for various studies such as LMP trend analysis, LMP prediction and load forecasting impact on LMP [7], etc.

IV. NUMERIC STUDY

The proposed method is illustrated on a revised IEEE 118-bus system. The system consists of 118 buses, 54 generators and 186 branches. System total load is 4242MW with 9966.2MW total generation capacity. The detailed system data are available in [12]. OPF related data are obtained from [7]. Specifically, five line limits are added into the transmission system: 345MW for line 69-77, 630MW for line 68-81, 106MW for line 83-85 and 94-100, 230MW for line 80-98. For illustration purpose, let the generators be sorted in the order of the bus numbers they are connected. The first 20 generators are assumed to be the cheapest generators with bids from \$10 to \$19.5 with \$0.5 increment; the next most expensive 20 generators have bids from \$30 to \$49 with a \$1 increment; and the remaining most expensive 14 generators have bids from \$70 to \$83 with \$1 increment. Besides all the above case changes, bus shunt admittance, branch charging capacitance and transformer tap ratios are all ignored in [7] because it deals with DC-OPF framework. For consistency and comparison purpose, this paper adopts the exact case changes as made in [7].

The whole spectrum of system load ranging from 0MW to 7686MW, i.e., from 0 to 1.81 p.u of base case load (PUBSL), is studied. This wide load range is selected only for the purpose of illustration and in practice load levels never go to such extremes. It should be noted that the total system load is distributed among load buses in proportion to base case bus load. Different load distribution pattern may lead to different LMP pattern. It should also be noted that the LMPs obtained from ACOPF model may be different from their counterparts of DCOPF model [6].

There are totally 57 CLLs identified using binary search algorithm with 0.001 PUBSL (or 4.242MW) precision. The CLLs are listed in Table 1.

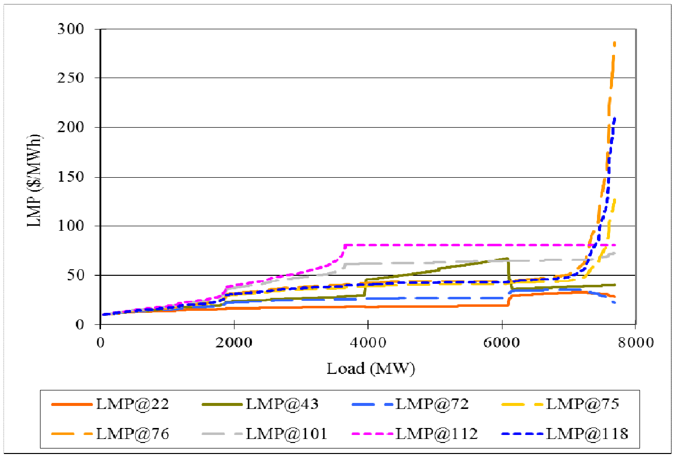
CLL(MW)	CLL(MW)	CLL(MW)	CLL(MW)
95.20	1654.86	3388.27	6076.67
189.74	1750.59	3431.78	6104.24
283.64	1833.82	3642.23	6125.51
376.90	1876.24	3752.35	6230.24
605.78	2007.12	3864.52	6779.09
755.89	2057.91	3944.60	6979.63
808.51	2338.11	4079.20	7189.89
904.93	2388.42	4434.54	7232.31
973.97	2537.58	4489.80	7335.75
1102.04	2729.31	4573.23	7450.30
1147.69	2910.65	4793.91	7513.82
1244.67	2983.60	5050.68	7611.12
1334.52	3076.90	5520.67	
1439.30	3139.73	5754.29	
1543.06	3193.33	6030.95	

Table 1: Identified CLLs in load range from 0MW to 7686MW.

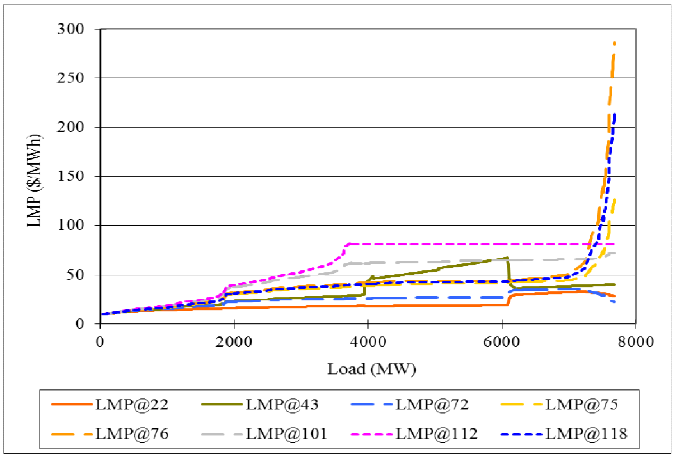
In order to obtain the actual LMP-Load curve, repetitive ACOPF runs have been performed from 42.42MW to 7686MW (i.e., 0.010 p.u. to 1.812 p.u. of base case load) with 4.242MW (i.e., 0.001 p.u. of base case load) sampling step. Thus, 1811 ACOPF have been solved. The ACOPF solutions have been fed into the linear curve-fitting process to generate the piece-wise linear model. It should be pointed out that the densely sampled load points are only for the purpose of generating the actual LMP-Load curve for benchmark purpose, instead of being required by linear curve-fitting process. In fact, as discussed in section III, when two load levels are chosen for each segment demarcated by any two immediately adjacent CLLs, only 116 ACOPF runs are actually needed.

The actual LMP versus Load curve for each of the 118 buses has already been shown in Figure 1. However, the figure is messy and hard to distinguish the curve for each individual bus. On the other hand, LMPs at some buses demonstrate similar pattern with respect to system load change. Therefore we can group LMPs into several categories, and extract only those with distinct curve patterns. In that regard, eight buses are selected whose LMPs are representative.

Figure 4(a) and 4(b) show respectively the actual LMP-Load curve and its piece-wise linear counterpart, the approximation curve obtained through the proposed method. The two set of curves look almost identical. One difference that may be appreciated by naked eyes is the LMP at bus 43 near 4000MW.



(a)



(b)

Fig. 4. LMP-Load curve of eight selected buses of the IEEE 118-bus system. (a) actual LMP-Load curve (b) piece-wise linear curve based on CLLs.

It is interesting to see that some portions of the LMP curves appear to be very serpentine in Figure 4 and therefore one would possibly pursue higher order polynomial or more sophisticated mathematical representation. Surprisingly, with the knowledge of CLLs, piece-wise linear model can suffice with high accuracy of fit. The curvy-looking segments in the LMP-load curves in Figure 4 are actually comprised of many smaller linear segments with different slopes. Each "small" segment demonstrates linear pattern very well if the curve is zoomed in for closer examination. It should be stressed that the key to a successful approximation is to determine break points of piece-wise linear curve, which relies on the identification of CLLs. Any other possible selection of break points may not yield such a good approximation model.

The majority of the differences between the actual LMP-Load curve and its piece-wise linear representation are very small. LMP may present step changes at some CLLs and slight change in load levels may result in significant change in LMP. On the other hand, the identified CLL can be slightly off from the actual CLL due to precision set for the binary search algorithm and limited length of computer word that relates to the precision to which numbers can be represented. Therefore, any approximation model may suffer from performance at CLLs or at load levels sufficiently close to CLLs. In that regard, we tolerate the price differences between the two curves at CLLs and a close vicinity of CLLs (e.g., 0.001pu, or, 4.2MW around the CLLs). Among totally 212636(=1802*118) prices in comparison, only 1582 prices (i.e., 0.74% in percentage) deviate the corresponding actual prices by more than 0.5$/MWh; only 27 prices deviate by more than $5/MWh; the largest price difference is $15.5/MWh. Percentage wise, only 31 prices deviate the corresponding actual prices by over 10%; the largest price difference in percentage is 22.2%.

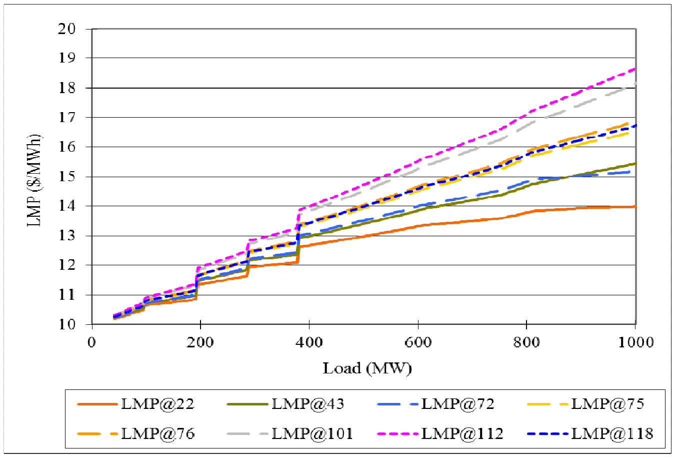
Overall, the piece-wise linear curve approximates the actual curve very well over the whole load spectrum by capturing every type of LMP variation pattern including step change pattern, polynomial pattern, linear pattern and more complex pattern. These patterns will be examined in more details in the rest of this section.

A. *Step Change Pattern of LMP*

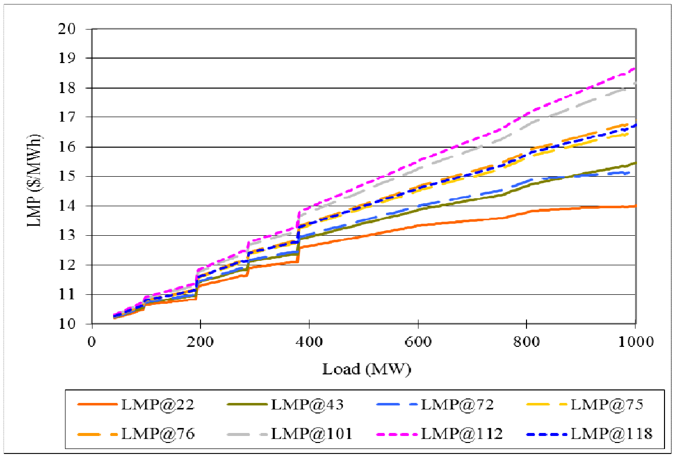
The zoomed-in graph of LMP-Load curves for load range 0MW to 1000MW is shown in Figure 5.

In Figure 5, the piece-wise linear curve tracks the actual curve very well, including the LMP jumps at certain load levels. For instance, LMPs at all eight buses jump up by $0.5/MWh at 189MW, which is an identified CLL. Before load grows to this level, there is no congestion and the generator at bus 4 is the only marginal unit. When load grows to 189MW, the marginal unit reaches capacity upper bound and becomes a non-marginal unit. At this point, the generator at bus 6 becomes the marginal unit. Since the new marginal unit is $0.5/MWh more expensive than the old marginal unit, the entire system will see a LMP uplift around $0.5/MWh.

Similar step change patterns are observed at other load levels and LMP change could be different than $0.5/MWh because of the loss effect.



(a)

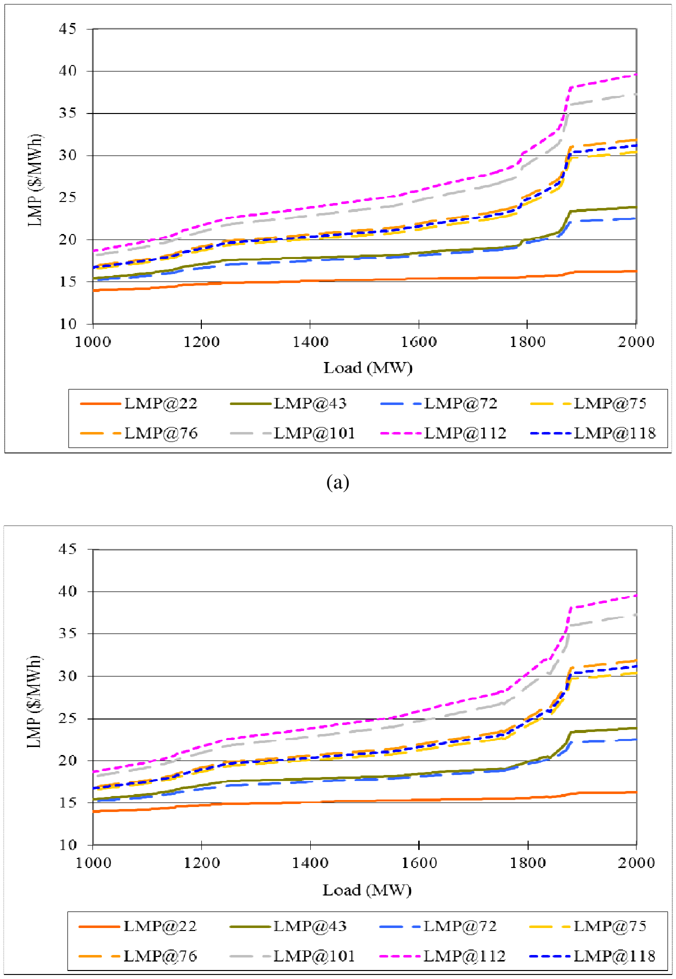


(b)

Fig. 5. LMP-Load curve of eight selected buses for load between 0MW and 1000MW. (a) actual LMP-Load curve (b) piece-wise linear curve based on CLLs.

B. Polynomial Pattern of LMP

In Figure 1, LMPs appear to be in a high order polynomial pattern (possibly quadratic polynomial pattern) around 1800MW. Therefore, the section for load range 1000MW to 2000MW has been magnified and shown in Figure 6.



(a)

(b)

Fig. 6. LMP-Load curve of eight selected buses for load between 1000MW and 2000MW. (a) actual LMP-Load curve (b) piece-wise linear curve based on CLLs.

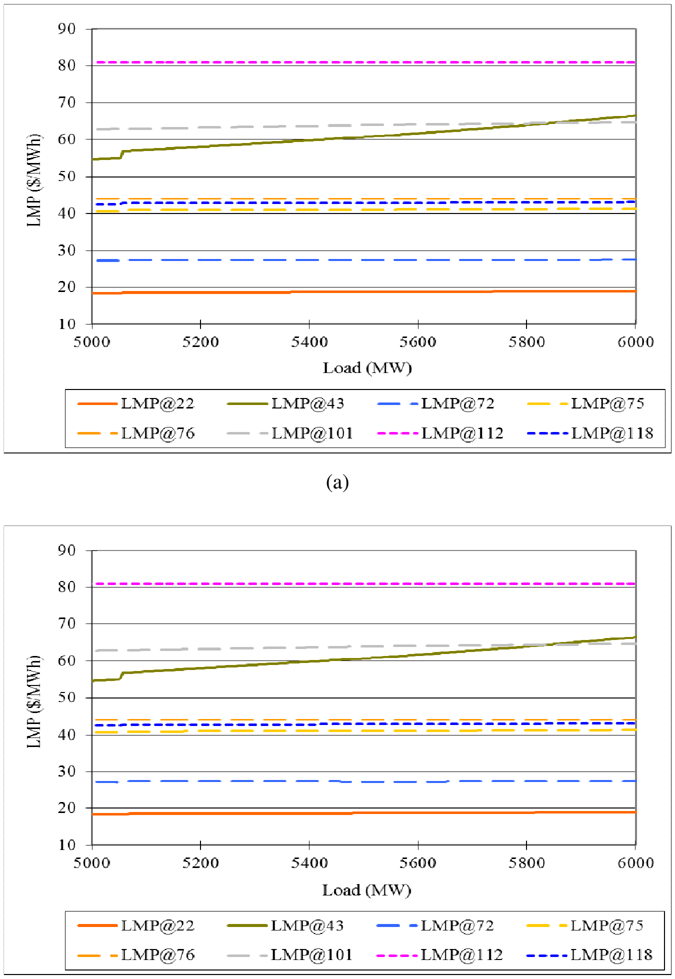
Figure 6 shows the seemly high order polynomial pattern around 1800MW which can be well approximated by a piece-wise linear curve with properly identified CLLs as the break points.

At a CLL, LMP may present a dramatic change such as a step change or a modest, continuous change. In Figure 6, the LMP curves change smoothly around CLL 1244MW where one more marginal unit occurs. It shows that a CLL does not necessarily indicate a LMP step change, rather, it is a good indicator of pattern change.

C. Linear Pattern of LMP

LMP-Load curves for load range 5000MW to 6000MW are shown in Figure 7.

Figure 7 shows LMP-Load curve may also demonstrate nearly perfect linear pattern over a broad range of load levels. A special case is that LMP keeps as constant. This typically happens for marginal unit buses. For instance, when load varies between 4000MW and 5000MW, generator at bus 112 is always a marginal unit, and therefore LMP at bus 112 keeps constant at $81/MWh, the bid of that generator.



(a)

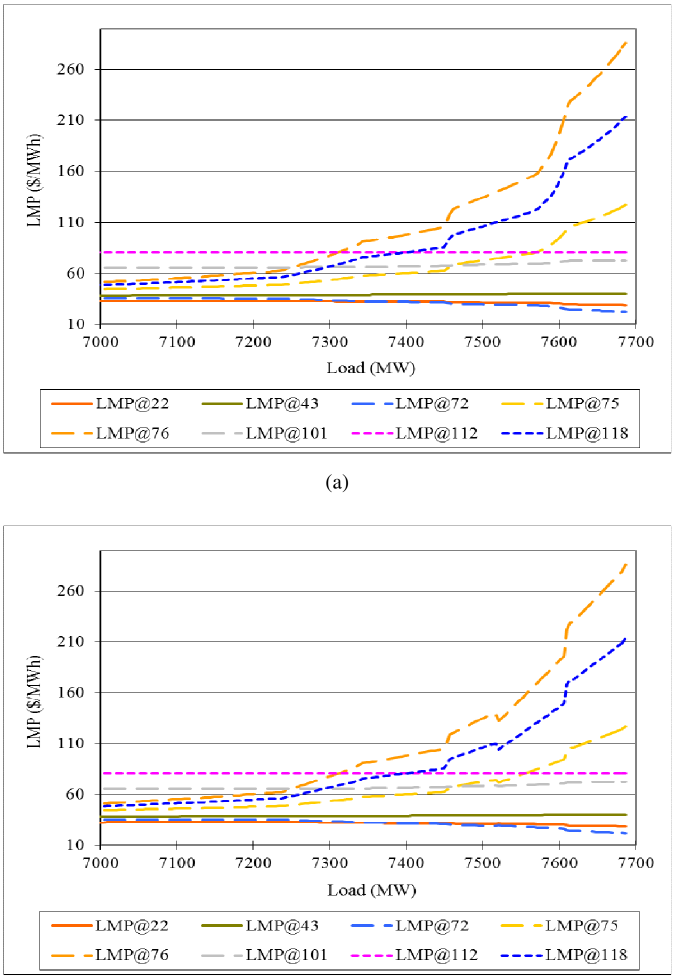
(b)

Fig. 7. LMP-Load curve of eight selected buses for load between 5000MW and 6000MW. (a) actual LMP-Load curve (b) piece-wise linear curve based on CLLs.

D. Complex Pattern of LMP

The LMP-Load curves for load range 7000MW to 7700MW are shown in Figure 8 where complex LMP pattern is observed. It is very encouraging to see the piece-wise linear model well represents the original serpentine curve when load is greater than 7300MW. Again, it attributes to the correct identification of CLLs so that any change in LMP pattern can be effectively appreciated.

Figure 8 shows that LMP can be much greater than the most expensive unit, which has been reported in existing literatures [11]. When load increases to extremely high level, all cheap units have reached or been close to the generation capacity. When there is congestion in the system, any incremental load increases may have to be served by expensive generators and it normally requires replacing cheap generation with more expensive generation to avoid transmission overloading.



(a)

(b)

Fig. 8. LMP-Load curve of eight selected buses for load between 7000MW and 7700MW. (a) actual LMP-Load curve (b) piece-wise linear curve based on CLLs.

V. CONCLUSIONS

In order to develop a mathematically simple model to represent the LMP-Load curve and capture all important curve patterns, this paper firstly reviews the concept of Critical Load Levels (CLLs) and approaches for locating the CLLs. Binary search algorithm is employed in this paper. Second, the identified CLLs are used to divide the LMP-Load curve into segments, and a linear curve-fitting approach is performed for each segment. Thus, a piece-wise linear model is established.

The proposed method has been tested on a revised IEEE 118-bus system. Results are very encouraging in that the piece-wise linear model not only closely represents the actual LMP-Load curve, but also captures various patterns of the curve including step changes pattern, linear pattern, higher order polynomial pattern and more complicated patterns.

It should be stressed that the main reason of the successful approximation lies in the utilization of the CLL concept and the near linearity of the curve between every two immediately adjacent CLLs. It is apparent that, for the same piece-wise linear curve-fitting approach, any alternative selection of break points other than CLL-based approach will not likely be able to produce such a good performance of approximation.

The obtained piece-wise linear model can greatly facilitate various analytical studies such as LMP trend analysis, LMP prediction and probabilistic LMP analysis.

Lastly, this paper has furthered the understanding of the CLL concept to be beyond the indicator of LMP step changes. Rather, in ACOPF framework, CLLs represent load levels where significant change of LMP pattern, not necessarily a step change, may occur.

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