# AN ORTHOGONAL DECOMPOSITION OF APPARENT POWER WITH APPLICATION TO AN INDUSTRIAL LOAD


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**Abstract - A complete characterization of reactive power is of increasing importance, given the abundance of nonlinear loads and distributed generators in modern power systems. This paper describes a physics-based methodology for decomposing the current (and consequently the apparent power) into mutually orthogonal components. Our framework also unifies several results from the literature, and enables useful descriptions of industrial loads in unbalanced, nonsinusoidal operation.**



***Keywords - Reactive Power, Polyphase Networks, Unbalanced Systems, Non-sinusoidal operation***


## 1 INTRODUCTION

STARTING with the work of Budeanu [1], many authors have aimed to characterize the concept of reactive power in the most general case, and to decompose the load current into physically meaningful mutually orthogonal components. The most detailed work to date appears to be that of Czarnecki [2], who introduced a decomposition consisting of five mutually orthogonal components. In this paper we introduce a new orthogonal decomposition that generalizes and refines the one proposed by Czarnecki, as well as those introduced by Sharon [3] and Shepherd and Zakikhani [4]. Our derivation uses only two fundamental mathematical concepts: 1) the Hilbert transform, and 2) orthogonal projection on a chain of nested subspaces, and treats the conductance and susceptance parts of the load admittance in a symmetric fashion. Moreover, we show that our *seven-component* orthogonal decomposition clearly relates Budeanu's ideas to Czarnecki's decomposition.

We use a Hilbert space terminology [5] to formulate our objectives and derive our results. Thus $v(t)$ and $i(t)$ are row vectors representing load voltage and current (nonlinear load, nonsinusoidal voltage source), which we view as elements in a Hilbert space of $n$-phase, square integrable, $T$-periodic waveforms, with the inner product defined by

$$\langle x , y \rangle \overset{\text{def}}{=} \frac{1}{T}\int_T x(t)y^\top(t)dt \qquad (1)$$

For instance, in this terminology the rms value of the polyphase voltage $v(t)$ is expressed as $\|v\| = \sqrt{\langle v , v \rangle}$, and the average (real) power delivered to the load is $P = \langle i , v \rangle$. Thus from the Cauchy-Schwarz inequality

$$P \le \|i\| \, \|v\| \overset{\text{def}}{=} S$$

we conclude that the apparent power $S$ is the highest average power delivered to the load, among all loads that have the same rms current $\|i\|$ (we assume that the voltage across the load remains unchanged as we change the apparent load seen by the source).

For a non-ideal load $P < S$, and this mismatch is the target of compensation devices (Fig. 1). The ideal (Fryze) compensator achieves

$$i_s(t) = i_F(t) \overset{\text{def}}{=} \frac{\langle i , v \rangle}{\|v\|^2}\, v(t) \qquad (2)$$

so that $P = \|i_F\| \, \|v\|$. The current difference $i_{exc}(t) \overset{\text{def}}{=} i(t) - i_F(t)$ is orthogonal to $i_F(t)$, so that $\|i\|^2 = \|i_F\|^2 + \|i_{exc}\|^2$. Our objective is to decompose $i_{exc}$ into several mutually orthogonal components that have a clear physical meaning.

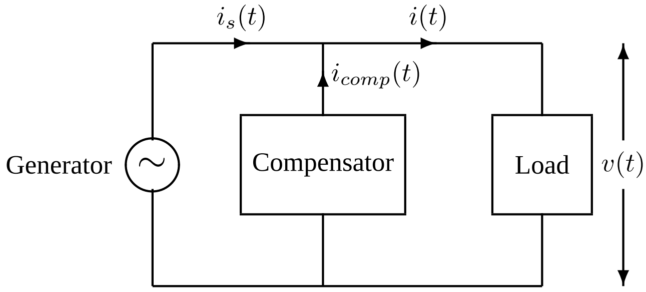


**Figure 1:** Load compensation in a power delivery system.

Our decomposition of the apparent power aims at load compensation, and in this context the physical limitations imposed on the compensator are a key factor. In most cases, the appropriate decomposition has 3 components, namely

$$i(t) = i_F(t) \oplus i_{nc}(t) \oplus i_c(t) \tag{3}$$

where $i_c(t)$ is the part of $i_{exc}(t)$ that can be compensated with the specific compensator under consideration, while $i_{nc}(t)$ is the part that is left uncompensated. The notation $\oplus$ denotes a sum of mutually orthogonal components. Indeed, many of the classical decompositions consist of such three components: Budeanu [1], Kusters and Moore [6], Shepherd and Zakikhani [4], Sharon [3], Czarnecki (single phase [7]), Depenbrock-Akagi-Nabae [9, 8]. However, in order to illustrate the precise relation among several distinct decompositions, we are led to construct a seven-component decomposition.

## 2 THE ORTHOGONAL DECOMPOSITION

We consider the space of signals that consists of $n$-phase, square-integrable, periodic signals. We assume that the voltage $v(t)$ is bandlimited with first $L$ harmonics being nonzero (and no DC component, for simplicity). Thus $v(t)$ consists of $nL$ mutually orthogonal components, i.e., $v(t) = \sum_{k=1}^{n} \sum_{\ell=1}^{L} v_{k,\ell}(t)$ where

$$\begin{aligned} v_{k,\ell}(t) &= 2\Re\{V_{k,\ell} e^{j\ell\omega t}\} e_k \\ e_k &\stackrel{\text{def}}{=} [\underbrace{0 \ldots 0\ 1}_{k}\ 0 \ldots 0] \end{aligned} \tag{4}$$

and $V_{k,\ell}$ is the phasor describing the $k$-th phase / $\ell$-th harmonic of $v(t)$, namely

$$V_{k,\ell} = \frac{1}{T} \int_T \left[ v(t) e_k^T \right] e^{-j\ell\omega t} dt$$

The collection of all waveforms, such as $v_{k,\ell}(t)$, that can be described by the expression (with fixed $k, \ell$)

$$2\Re\{C e^{j\ell\omega t}\} e_k$$

for some (complex) phasor $C$ is a two-dimensional subspace of our Hilbert space. Every one of its members can be obtained by a linear combination of the two mutually orthogonal waveforms $(\cos \ell\omega t)\ e_k$ and $(\sin \ell\omega t)\ e_k$.

An alternative orthogonal basis for the subspace of all $k$-th phase / $\ell$-th harmonic waveforms is given by the two waveforms $\{v_{k,\ell}(t), \mathcal{H}v_{k,\ell}(t)\}$, where $\mathcal{H}x(t)$ denotes the Hilbert transform [10] of a waveform $x(t)$. For periodic waveforms this linear transform is completely determined by

$$\mathcal{H}\{e^{j\ell\omega t}\} = -j(\operatorname{sgn} \ell) e^{j\ell\omega t} \quad (\omega > 0)$$

so that

$$\mathcal{H}\{\cos \ell\omega t\} = \sin \ell\omega t, \quad \mathcal{H}\{\sin \ell\omega t\} = -\cos \ell\omega t$$

Two key properties of the Hilbert transform (for every periodic waveform $x(t)$ with no DC component) are $\|\mathcal{H}x\| = \|x\|$ and $\langle\ \mathcal{H}x\,,\, x\ \rangle = 0$, so that indeed $\{v_{k,\ell}(t), \mathcal{H}v_{k,\ell}(t)\}$ form an orthogonal basis for the subspace of all $k$-th phase / $\ell$-th harmonic waveforms. Consequently, the waveforms $\{[v_{k,\ell}(\cdot),\ \mathcal{H}v_{k,\ell}(\cdot)]; 1 \le k \le n, 1 \le \ell \le L\}$ form an orthogonal basis for the subspace of all $n$-phase signals that are bandlimited to $L$ harmonics. This was observed also by Mathis and Marten [11] in the single-phase non-sinusoidal case. If the load is nonlinear, the current may have additional harmonics, so that

$$i(t) = i_{\|}(t) \oplus i_{\perp}(t)$$

where the "inband" component $i_{\|}(t)$ corresponds to the harmonics $1 \le \ell \le L$, and the "out-of-band" component corresponds to the remaining harmonics. If the current $i(t)$ has a DC component (but the voltage does not), then $i_{DC}$ is part of $i_{\perp}(t)$. On th other hand, if $v(t)$ has a DC component, then our orthogonal basis has to be augmented by the (constant) single-phase waveforms $\Phi_k(t) = e_k$ (for all $t$). The augmented basis spans the $n(2L+1)$ dimensional subspace of all $n$-phase periodic waveforms that are limited to $L$ harmonics *and* may have a DC component. Since voltages in large power systems rarely have a DC component, and in the interest of brevity, we continue to assume in the sequel that $v(t)$ has no DC component.

Using the orthogonal basis $\{v_{k,\ell}(\cdot),\ \mathcal{H}v_{k,\ell}(\cdot); 1 \le k \le n, 1 \le \ell \le L\}$ we can decompose $i_{\|}(t)$ into its $2nL$ orthogonal components, viz.,

$$i_{\|}(t) = \underbrace{\sum_{k,\ell} g_{k,\ell}\ v_{k,\ell}(t)}_{i_v(t)} + \underbrace{\sum_{k,\ell} b_{k,\ell}\ \mathcal{H}v_{k,\ell}(t)}_{i_w(t)} \tag{5}$$

where (recall that $\|\mathcal{H}x\| = \|x\|$)

$$g_{k,\ell} \stackrel{\text{def}}{=} \frac{\langle\ i\,,\, v_{k,\ell}\ \rangle}{\|v_{k,\ell}\|^2}, \qquad b_{k,\ell} \stackrel{\text{def}}{=} \frac{\langle\ i\,,\, \mathcal{H}v_{k,\ell}\ \rangle}{\|v_{k,\ell}\|^2}, \tag{6}$$

The relation between $v(t)$ and $i_{\|}(t)$ can be interpreted in terms of an *equivalent linear load* with admittance $g_{k,\ell} - jb_{k,\ell}$ . For a nonlinear load the values of $(g_{k,\ell},\ b_{k,\ell})$ depend on the load voltage, but for a linear load $g_{k,\ell} - jb_{k,\ell}$ is a voltage-independent constant.

The orthogonal decomposition (5) provides a two-component orthogonal decomposition of $i_{\|}(t)$, viz.,

$$i_{\|}(t) = i_v(t) \oplus i_w(t) \tag{7}$$

Thus $i_v \in \mathcal{V} \stackrel{\text{def}}{=} \mathtt{span}\{v_{k,\ell}\,;\ 1 \le k \le n, 1 \le \ell \le L\}$ and $i_w \in \mathcal{W} \stackrel{\text{def}}{=} \mathtt{span}\{\mathcal{H}v_{k,\ell}\}$.

In view of (2), $i_F(t)$ is the orthogonal projection of the load current $i(t)$ on the one-dimensional subspace $\mathtt{span}\{v(\cdot)\} \subset \mathcal{V}$. This means that

$$i_g(t) \stackrel{\text{def}}{=} i_v(t) - i_F(t)$$

is orthogonal to $i_F(t)$, so that we obtain the orthogonal decomposition

$$i_v(t) = i_F(t) \oplus i_g(t), \quad i_g(t) = \sum_{k,\ell} (g_{k,\ell} - \mu_g)\; v_{k,\ell}(t) \tag{8}$$

Here

$$\mu_g \stackrel{\text{def}}{=} \sum_{k,\ell} p_{k,\ell}\; g_{k,\ell}$$

is a weighted mean of the load conductances $\{g_{k,\ell}\}$, where $p_{k,\ell}$ are (the voltage square) normalized weights

$$p_{k,\ell} \stackrel{\text{def}}{=} \frac{\|v_{k,\ell}\|^2}{\|v\|^2} = \frac{\|v_{k,\ell}\|^2}{\sum_{k,\ell} \|v_{k,\ell}\|^2}$$

Thus $i_F(t) = \mu_g v(t)$ is associated with the weighted mean of $g_{k,\ell}$ , while $i_g(t)$ is associated with the spread of $g_{k,\ell}$ in both phase and frequency around $\mu_g$ .

Indeed, the norm of $i_g(t)$ is directly related to the standard deviation of $\{g_{k,\ell}\}$, viz.,

$$\|i_g\|^2 = \sum_{k,\ell} (g_{k,\ell} - \mu_g)^2 \|v_{k,\ell}\|^2 = \sigma_g^2 \|v\|^2$$

where

$$\sigma_g \stackrel{\text{def}}{=} \sqrt{\sum_{k,\ell} p_{k,\ell} (g_{k,\ell} - \mu_g)^2} \tag{9}$$

In the special case when $g_{k,\ell} = {}^1\!/_R$ for all $k, \ell$, e.g. for a purely resistive balanced load, we get $\sigma_g = 0$, and thus $i_g(t) \equiv 0$.

Given that the Fryze current $i_F(t)$ is the orthogonal projection of $i(t)$ on $v(t)$ [5], we now consider the projection of $i(t)$ on $\mathcal{H}v(t)$ and denote it as the "Budeanu current"

$$i_B(t) \stackrel{\text{def}}{=} \frac{\langle\, i\,,\, \mathcal{H}v\,\rangle}{\|v\|^2} \mathcal{H}v(t) = \mu_b\, \mathcal{H}v(t) \tag{10}$$

where we used the fact that, from (5), $\langle\, i\,,\, \mathcal{H}v\,\rangle = \sum_{k,\ell} b_{k,\ell} \|\mathcal{H}v_{k,\ell}\|^2$, and we defined the weighted mean susceptance

$$\mu_b \stackrel{\text{def}}{=} \sum_{k,\ell} p_{k,\ell}\; b_{k,\ell}\,. \tag{11}$$

The associated power component is $|Q_B| = \|i_B\|\,\|v\|$, where

$$\begin{aligned} Q_B &\stackrel{\text{def}}{=} \langle\, i\,,\, \mathcal{H}v\,\rangle \\ &= \sum_{k,\ell} 2|I_{k,\ell}||V_{k,\ell}| \sin\Big(\arg(V_{k,\ell} I_{k,\ell}^*)\Big) \end{aligned}$$

which is a polyphase extension of Budeanu's definition of reactive power. The analogy with the Fryze current and real power is now complete, as $i_B(t)$ is the orthogonal projection of the load current $i(t)$ on the one-dimensional subspace $\mathtt{span}\{\mathcal{H}v(\cdot)\} \subset \mathcal{W}$ and, therefore,

$$i_b \stackrel{\text{def}}{=} i_w(t) - i_B(t)$$

is orthogonal to $i_B(t)$, so that we obtain the orthogonal decomposition

$$i_w(t) = i_B(t) \oplus i_b(t)\,, \quad i_b(t) = \sum_{k,\ell} (b_{k,\ell} - \mu_b) \mathcal{H}v_{k,\ell}(t) \tag{12}$$

Again, the norm of $i_b(t)$ is directly related to $\sigma_b$ , the standard deviation of $\{b_{k,\ell}\}$, viz.,

$$\|i_b\| = \sigma_b \|v\|, \quad \sigma_b \stackrel{\text{def}}{=} \sqrt{\sum_{k,\ell} p_{k,\ell} (b_{k,\ell} - \mu_b)^2} \tag{13}$$

The special case $\sigma_b = 0$ is relatively rare (as compared with $\sigma_g = 0$), since it requires that the imaginary part of the load admittance be independent of phase *and of frequency*. This requirement is violated by linear inductive and capacitive loads.

In summary, we have the 4-component orthogonal decomposition of the inband current

$$i_{\|}(t) = \underbrace{i_F(t) \oplus i_g(t)}_{i_v(t)} \oplus \underbrace{i_B(t) \oplus i_b(t)}_{i_w(t)} \tag{14}$$

We can further refine our current decomposition to provide an indication whether the load is balanced or not among phases. We shall call the load G-balanced (respectively B-balanced) when $g_{k,\ell}$ (resp. $b_{k,\ell}$) is independent of the phase index $k$. For a G-balanced load

$$i_v(t) \in \mathtt{span}\{\sum_{k=1}^{n} v_{k,\ell}(t)\,;\ \ 1 \le \ell \le L\} \subset \mathcal{V}$$

so, in general, we can refine the orthogonal decomposition (8) of $i_v(t)$ using the orthogonal projection of the load current $i(t)$ on this subspace. This leads to

$$i_v(t) = i_F(t) \oplus \underbrace{i_{gs}(t) \oplus i_{gu}(t)}_{i_g(t)} \tag{15}$$

where

$$i_{gu}(t) \stackrel{\text{def}}{=} \sum_{k,\ell} \Big[ g_{k,\ell} - \mu_g(\ell) \Big] v_{k,\ell}(t)$$

is the current corresponding to the unbalanced portion of the load conductance, and

$$i_{gs}(t) = i_g(t) - i_{gu}(t) = \sum_{k,\ell} \Big[ \mu_g(\ell) - \mu_g \Big] v_{k,\ell}(t)$$

is the remainder of $i_g$, due to the spread of $\{g_{k,\ell}\}$ with frequency. Here $\mu_g(\ell)$ is a frequency-dependent weighted mean

$$\mu_g(\ell) \stackrel{\text{def}}{=} \frac{\langle\, i\, ,\, \sum_k v_{k,\ell}(t)\, \rangle}{\| \sum_k v_{k,\ell} \|^2} = \frac{\sum_k g_{k,\ell} \| v_{k,\ell} \|^2}{\sum_k \| v_{k,\ell} \|^2}$$

The norms of $i_{gs}(\cdot), i_{gu}(\cdot)$ satisfy

$$\| i_{gs} \| = \hat{\sigma}_g \| v \|, \quad \| i_{gu} \| = \check{\sigma}_g \| v \| \tag{16}$$

where $\check{\sigma}_g$ is the standard deviation with respect to phase, averaged over all frequencies, viz.,

$$\check{\sigma}_g \stackrel{\text{def}}{=} \sqrt{\sum_{k,\ell} p_{k,\ell} | g_{k,\ell} - \mu_g(\ell) |^2} \tag{17}$$

and

$$\hat{\sigma}_g^2 \stackrel{\text{def}}{=} \sigma_g^2 - \check{\sigma}_g^2 = \sum_{k,\ell} p_{k,\ell} | \mu_g(\ell) - \mu_g |^2 \tag{18}$$

describes the variation across frequency. When the load is G-balanced, $i_{gu} = 0$.

Next, we decompose $i_b(t)$ in the same way, namely $i_b(t) = i_{bs}(t) \oplus i_{bu}(t)$ to finally obtain the seven-component decomposition

$$i = i_F \oplus i_{gs} \oplus i_{gu} \oplus i_B \oplus i_{bs} \oplus i_{bu} \oplus i_\perp \tag{19}$$

Note that we can define only two signed power components using the projections of the current on $v$ and $\mathcal{H}v$, respectively:

$$P = \mu_g \| v \|^2 = \langle\, i_F\, ,\, v\, \rangle = \langle\, i\, ,\, v\, \rangle \tag{20}$$

$$Q_B = \mu_b \| v \|^2 = \langle\, i_B\, ,\, \mathcal{H}v\, \rangle = \langle\, i\, ,\, \mathcal{H}v\, \rangle \tag{21}$$

where $P$ is the real (active) power and $Q_B$ is the Budeanu reactive power. Since $\mu_g > 0$, we have $P = \| i_F \| \; \| v \|$, but $Q_B$ need not be positive, so $|Q_B| = \| i_B \| \; \| \mathcal{H}v \| = \| i_B \| \; \| v \|$.

We can express the orthogonal current decomposition (19) also in terms of powers (the orthogonality among various currents implies the possibility to define powers by calculating squares of norm of currents, and multiplying with the square of the voltage norm throughout)

$$S^2 = P^2 + N_s^2 + N_u^2 + Q_B^2 + Q_s^2 + Q_u^2 + S_\perp^2 \tag{22}$$

The standard nomenclature exists only for some components, e.g. $P$ and $Q_B$. We denote $N_s$ and $N_u$ as spread (over frequencies) and unbalanced (over phases) co-active powers, respectively, and $Q_s$ and $Q_u$ as spread and unbalanced co-reactive powers, respectively. We propose to use the term *reactive power* to collectively denote the three susceptance-dependent terms, namely $\sqrt{Q_B^2 + Q_s^2 + Q_u^2}$. Similarly, the overall co-active power is $N_g = \sqrt{N_s^2 + N_u^2}$. The last term, $S_\perp$, is the out-of-band apparent power.

## 3 RELATIONSHIPS WITH EXISTING DECOMPOSITIONS

Our two-component decomposition (5) reduces, in the single-phase case, to that of Shepherd and Zakikhani [4], as shown in the following Table (with $S_v = \|v\|\|i_v\|$ and $Q_w = \|v\|\|i_w\|$):

| Shepherd & Zakikhani | $S_R$ | $S_X$ |
|---|---|---|
| Ours | $i_v,\ S_v$ | $i_w,\ Q_w$ |

If we split $i_v(t)$, but not $i_w(t)$, viz.,

$$i_\| = i_F(t) \oplus i_g(t) \oplus i_w(t)$$

we obtain the decomposition of Sharon, viz.

| Sharon | $P$ | $S_c$ | $S_Q$ |
|---|---|---|---|
| Ours | $i_F, P$ | $i_g,\ N_g$ | $i_w,\ Q_w$ |

The relation to Czarnecki's decomposition [2] is detailed below

| Czarnecki | $i_a$ | $i_s$ | $i_r$ | $i_u$ | $i_g$ |
|---|---|---|---|---|---|
| Ours | $i_F$ | $i_{gs}$ | $i_B \oplus i_{bs}$ | $i_{gu} \oplus i_{bu}$ | $i_\perp$ |

Note that Czarnecki does not indicate that Budeanu's current (resp. power) is a subcomponent of his current $i_r(t)$ (resp. $Q_r$), and he combines the unbalanced components corresponding to conductances and susceptances.

A polyphase generalization of the Kusters and Moore decomposition [6] can also be obtained: it requires a different decomposition of our $i_w(t)$, and one that is *phase-dependent*. The decomposition used for the $k$-th phase of $i_w(t)$ is based on the subspace $\mathtt{span}\{\sum_\ell \ell^r\ [\mathcal{H}v_{k,\ell}(t)]\}$, where either $r = 1$ (for capacitors) or $r = -1$ (for inductors). Note that this subspace is one-dimensional (since the phase index $k$ has a fixed value), but not colinear with the (also one-dimensional) "Budeanu subspace" $\mathtt{span}\{\sum_l [\mathcal{H}v_{k,\ell}(t)]\}$.

## 4 BASIC EXAMPLES

In this section we study general conditions under which certain power components vanish. In the case of balanced loads, $i_{gu}$ and $i_{bu}$ vanish, while in the case of balanced *and* frequency-independent loads the same happens to $i_g$ and $i_b$. In the case of a purely resistive (reactive, respectively) load, we have $i_w \equiv 0$ (or $i_v \equiv 0$, resp.)

**Example 1** – consider the case of a three phase system, with balanced voltages (of unit magnitude) and with a resistive load in phase *a*, a purely inductive one in phase *b* and a purely capacitive one in phase *c*, each drawing a current of unit magnitude.

In this (sinusoidal) case $\|i\|^2 = \|v\|^2 = 3$, and $g_{11} = 1,\ g_{21} = 0 = g_{31},\ b_{11} = 0,\ b_{21} = 1$ and $b_{31} = -1$. There is only one frequency, so $\mu_g(\ell) \equiv \mu_g, \quad \mu_b(\ell) \equiv \mu_b$ and, therefore, $i_{gs} = 0 = i_{bs}$. Also $\|v_1\| = \|v_2\| = \|v_3\| = 1$, so that $\mu_g = 1/3,\ \sigma_g^2 = 2/9,\ \ \mu_b = 0,\ \sigma_b^2 = 2/3$, and

$$\|i_F\|^2 = \mu_g^2\|v\|^2 = \frac{1}{3},\ \ i_B = 0,$$

$$\|i_g\|^2 = \sigma_g^2\|v\|^2 = \frac{2}{3},\ \ \|i_b\|^2 = \sigma_b^2\|v\|^2 = 2$$

In terms of powers, $S^2 = 9, \quad P^2 = 1, \quad N_u^2 = 2, \quad Q_u^2 = 6$. The reactive power is entirely due to the imbalance of the susceptances, but there is also a significant co-active power, due to the imbalance of the conductances.

**Example 2** – consider the case of a single-phase, ideal linear inductor (of unit reactance), driven by a fundamental (of unit magnitude) and the fifth harmonic (of magnitude 0.1) so that $\|V_1\| = 1,\ \|V_5\| = 0.1,\ \ g_\ell = 0$ for all $\ell$, and $b_1 = 1, \quad b_5 = 0.2$. Thus $\|v\|^2 = 1.01,\ \mu_b = 0.9921$, and $\sigma_b = 0.0792$. Since there is only one phase, all unbalanced components are, of course, identically zero, but, interestingly enough, Budeanu's power does not capture all of the reactive power. Indeed, $i(t) = i_w(t) = i_B(t) \oplus i_b(t)$, where $\|i_B\| = \mu_B\|v\| = 0.9970$ and $\|i_b\| = \sigma_b\|v\| = 0.0796$, and so $\|i\|^2 = \|i_B\|^2 + \|i_b\|^2 = 1.0004$. Finally, $S^2 = 1.0104$, $Q_B^2 = 1.0040$, and the (spread) co-reactive power is $Q_b^2 = 0.0064$. While the numerical discrepancy is small, it is important to note that a difference will exist whenever the load is frequency dependent.

**Example 3** – this example was used in [5] to demonstrate that under certain conditions, a reactive power compensator based on instantaneous reactive power [8] may inject harmonics that are absent in both voltage and current. A two phase resistive load has voltages

$$v(t) = \begin{bmatrix} V_1\cos(\omega t) \\ V_2\sin(\omega t) \end{bmatrix}^T,\ \ i(t) = \begin{bmatrix} P_0/V_1\cos(\omega t) \\ P_0/V_2\sin(\omega t) \end{bmatrix}^T$$

Note the particular choice of (constant) parameters – the resistance in each phase is proportional to the square of the voltage magnitude in that phase. From $\|v\|^2 = (V_1^2+V_2^2)/2$, $\|i\|^2 = P_0^2/2(1/V_1^2 + 1/V_2^2)$, $g_{11} = P_0/V_1^2$, $g_{21} = P_0/V_1^2$, and all susceptance-related and out-of-band terms identically zero, we get

$$\mu_g = \frac{2P_0}{V_1^2+V_2^2},\quad \sigma_g^2 = \frac{P_0^2}{4\|v\|^4}\Big(\frac{V_1}{V_2} - \frac{V_2}{V_1}\Big)^2$$

Finally, $P = \mu_g\|v\|^2 = P_0$ and $N_g = \frac{P_0}{2}|\frac{V_1}{V_2} - \frac{V_2}{V_1}|$ For $V_1 = 1,\ V_2 = 0.1,\ P_0 = 1$, we get $\|v\|^2 = 0.505$, $\|i\|^2 = 50.5$, and $g_{11} = 1,\ g_{21} = 100$. There is again only one frequency, so $\mu_g(\ell) \equiv \mu_g = 1.98$, $\sigma_g^2 = 96.08$, and $\|i_F\|^2 = \mu_g^2\|v\|^2 = 1.98$ , $\|i_g\|^2 = \sigma_g^2\|v\|^2 = 48.52 = \|i_{gu}\|^2$ since there is no frequency spread. In terms of powers, $S^2 = 25.5,\ P^2 = P_0^2 = 1,\ N_u^2 = 24.5$, and all other terms are zero. Note that this is more intuitive in this *purely resistive* circuit than the instantaneous *reactive* power [8] that equals $q(t) = \frac{P_0}{2}(\frac{V_1}{V_2} - \frac{V_2}{V_1})\sin(2\omega t)$.

## 5 INDUSTRIAL EXAMPLE - ADJUSTABLE SPEED DRIVE

We apply our decomposition to the example of a 350-HP industrial adjustable speed drive (ASD), operating in parallel with a linear unbalanced load. The load voltages are mildly nonsinusoidal and unbalanced, as seen in Fig. 2; the corresponding current waveforms (with large 5-th and 7-th harmonic) are shown in Fig. 3.

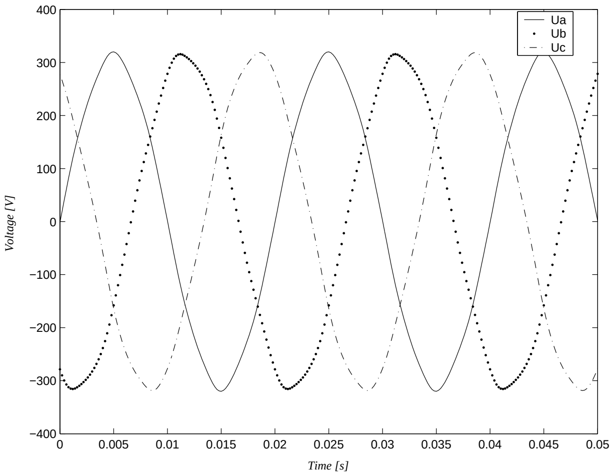


**Figure 2:** Polyphase industrial load - voltages.

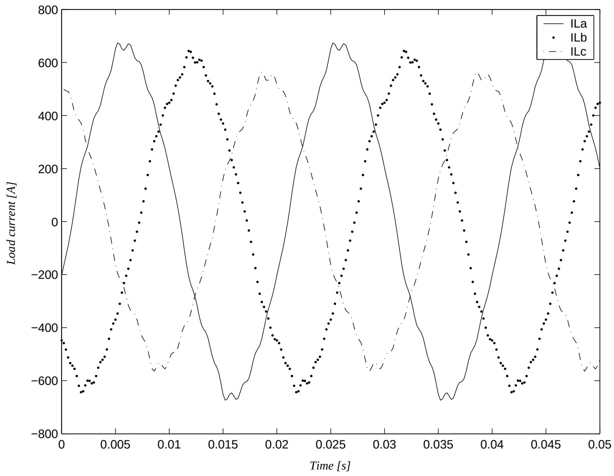


**Figure 3:** Polyphase load - currents.

The voltages contain (odd) harmonics up to eleventh, while current has harmonics up to 19-th. Representative values are given in Tables 1 and .

| | *Phase 1* | | *Phase 2* | | *Phase 3* | |
|---|---|---|---|---|---|---|
| *Order* | *Mag.* $[V]$ | *Phase* $[°]$ | *Mag.* $[V]$ | *Phase* $[°]$ | *Mag.* $[V]$ | *Phase* $[°]$ |
| 1 | 221.4 | -0.18 | 222.9 | -119.6 | 224.2 | 119.8 |
| 3 | 0.56 | 109.4 | 0.56 | -130.6 | 0.56 | -10.56 |
| 5 | 3.74 | -142.6 | 4.25 | -27.68 | 4.32 | 100.55 |
| 7 | 1.243 | 26.63 | 1.69 | -127.8 | 0.784 | 95.36 |
| 9 | 0.225 | -126.6 | 0.225 | -6.62 | 0.225 | 113.4 |
| 11 | 0.404 | 54.43 | 0.404 | 174.4 | 0.404 | -65.57 |

**Table 1:** Load voltage harmonics.

| *Order* | *Mag.* $[A]$ | *Phase* $[°]$ |
|---|---|---|
| 1 | $342.8$ | $-13$ |
| 3 | $3.48$ | $33$ |
| 5 | $13.37$ | $-16$ |
| 7 | $7.20$ | $77$ |
| 9 | $0.34$ | $49$ |
| 11 | $4.46$ | $145$ |
| 13 | $2.06$ | $-164$ |
| 15 | $0.34$ | $-45$ |
| 17 | $7.20$ | $-45$ |
| 19 | $5.14$ | $-51$ |

**Table 2:** Load current harmonics in phase 1 of ASD.

The overall load consists of the adjustable speed drive and of additional unbalanced three phase current sink:

$$i_{sink,1} = 120\sin(2\pi\,50\,t - 30°)A;$$

$$i_{sink,2} = 90\sin(2\pi\,50\,t - 120° + 10°)A;$$

$$i_{sink,3} = 40\sin(2\pi\,50\,t + 120° - 45°)A;$$

Then we calculate

$$\begin{aligned} \mu_g &= 1.8283[1/\Omega];\ \ \mu_b = 0.4543[1/\Omega] \\ \hat{\sigma}_g &= \frac{\|i_{gs}\|}{\|v\|} = 0.0310[1/\Omega] \\ \check{\sigma}_g &= \frac{\|i_{gu}\|}{\|v\|} = 0.1614[1/\Omega] \\ \hat{\sigma}_b &= \frac{\|i_{bs}\|}{\|v\|} = 0.0313[1/\Omega] \\ \check{\sigma}_b &= \frac{\|i_{bu}\|}{\|v\|} = 0.1399[1/\Omega] \end{aligned}$$

and $\|v\| = 386.1V$; $\|i\| = 732.39A$; $\|i_F\| = 705.88A$, $\|i_B\| = 175.40A$; $\|i_{gs}\| = 11.95A$; $\|i_{gu}\| = 62.33A$, $\|i_{bs}\| = 12.12A$; $\|i_{bu}\| = 54.0A$

The corresponding decomposition of the reactive power is shown in Table 3.

| $S$ | $P$ | $N_s$ | $N_u$ | $Q_B$ | $Q_s$ | $Q_u$ | $S_\perp$ |
|---|---|---|---|---|---|---|---|
| 282 770 | 272 540 | 4 616 | 24 065 | 67 720 | 4 679 | 20 868 | 6 078 |

**Table 3:** Powers in the adjustable speed drive plus unbalanced fundamental frequency load example.

Note that components corresponding to phase unbalance are larger than frequency spread and out-of-band components. If only the adjustable speed drive is connected and the voltages are balanced, the unbalanced power components are zero, as expected (see Table 4).

| $S$ | $P$ | $N_s$ | $N_u$ | $Q_B$ | $Q_s$ | $Q_u$ | $S_\perp$ |
|---|---|---|---|---|---|---|---|
| 229 654 | 223 177 | 12 480 | 0 | 51 302 | 9 976 | 0 | 6 776 |

**Table 4:** Powers in the adjustable speed drive example with balanced nonsinusoidal voltages.

## 6 CONCLUSIONS

In this paper we described a methodology for decomposing the apparent power into mutually orthogonal components. Our decomposition places no limitations on the number of phases or harmonics, and allows for arbitrary unbalance among phases and over frequencies. This may prove beneficial for characterization of modern power systems, as the emergence of nonlinear loads and distributed generation results in both complicated power flow patterns and in the need to precisely characterize them. The measurement requirements for applying our decomposition in industrial practice are not more demanding than in previously proposed decompositions, and are typically satisfied in modern industrial plants. In addition to metering purposes, the methodology may contribute towards equipment design and control, as it builds on common engineering intuition by utilizing physics-based properties to characterize the energy flows.

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