FAILURE DISTRIBUTION OF REPAIRABLE UNITS APPROXIMATION THROUGH MARKOV CHAINS

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Abstract - Failure distributions of components and assemblies are commonly approximated by analytical functions, e.g. by exponential or Weibull distributions. The advantages of this approach are its simplicity and the staightforwardness of the calculations. Another possibility is to describe failure distributions by means of Markov or semi-Markov chains, however several constraints are thereby introduced. The present paper deals with the use of Markov chains as an approximation of failure distributions of repairable units and assemblies due to ageing, enlighting the main constraints, advantages and disadvantages as well as proposing some guidelines for the use of such approaches.

Keywords - failure distribution, Markov chains, ageing modelling, failure simulation

1 INTRODUCTION

Failure distributions of components and assemblies can be approximated by means of analytical functions (e.g. through an exponential or Weibull distribution). This approach is well established in many fields of engineering and its advantages are mainly the simplicity and the straightforwardness of the calculations. On the other hand, Markov and semi-Markov chains are in wide use in the area of stochastics and signal processing. Markov chains are also commonly used in reliability calculations, however with rigorous constraints. The main constraint introduced by the use of time-homogeneous Markov models relates to the constant transition (or failure) rates between the different states, leading to inadequate modelling of ageing phenomena, where the failure rates depend from the history.

In this paper, investigations are carried out regarding the use of time-homogeneous Markov chains for failure distribution approximation in the case of components' assemblies whose failure distributions depend on ageing. Several reductions have been made:

- An assembly of components, rather than merely one component, was taken as object of investigation, this fact resulting to an important attribute of the model: As only one component of the assembly can fail at each time, and this component may either be repaired or replaced, it is assumed that after the repair (or replacement) the assembly returns to exactly the same state it was before the failure.
- The residence time at the fault state, or correspondingly, the duration of the repair, is assumed to be negligible related to the residence time at one state.
- For the investigations two types of failure distributions have been modelled: at a first step the Weibull distribution was taken as a reference, as it describes closely the failure distribution of ageing mechanical components. At a second step, a real failure distribution of SF_6 circuit breakers was modelled.

The present paper is organised as follows: in section 2 the applicability of Markov chains is discussed on the basis of some of their main attributes, then in section 3 the application of Markov chains for the approximation of failure distributions due to ageing is exemplary performed for the two different test sets. After a short examination of the empirical applicability, some improvements are proposed based on statistical analysis of the distributions to be modelled in section 4. Finally, in the last paragraphs some conclusions are drawn.

2 FUNDAMENTALS OF MARKOV CHAINS

The definition as well as some major attributes of Markov chains with finite number of states may be found in the appendix. Here the general applicability of Markov chains for modelling failure distributions due to ageing will be discussed.

A finite Markov chain can be represented by its transitions matrix **P** where the element p_{ij} is the probability of transition from state i to state j at the n^{th} transition.

$$p_{ij}(n) = \Pr\left(X_n = j \,|\, X_{n-1} = j\right) \qquad (1)$$

According to equation 1 two cases may be distinguished:

- p_{ij} independant of n: The transition matrix **P** is constant and independant of the residence time in each state. The Markov chain is therefore *(time) homogeneous*.
- $p_{ij} = f(n)$: The probability of transition from state i to state j depends on the transition time n, leading to a *time-variant* Markov chain.

A property of Markov chains which is of great importance for stochastic applications is *ergodicity*. An ergodic finite Markov chain has a steady-state probability vector, say π, to which $\pi\left(n\right)$ tends as $n \rightarrow \infty$. This steady-state is independant of starting probabilities $\pi(0)$. There are two simple conditions that together ensure that a finite Markov chain is ergodic:

- all the states of the chain must be intercommunicating, i.e. for any two states i and j, it must be possible to get from i to j, although not necessarily in one step.

- the states must not be cyclic, i.e. there are no states that can only be visited at regularly spaced intervals.

From the above conditions it is obvious that a Markov chain with absorbing states (states described by the transition probability $p_{ii} = 1$) cannot be ergodic.

For the representation of failure distributions over time following assumptions are made:

- the components under investigation are considered to be assemblies rather than seperate units. This meets the requirement that a component normally has different failure modes.
- the duration for repair is negligible, as the process describes the deterioration process, which may ammount decades.
- failures can be classified into two groups: age-dependant (due to internal deterioration) and age-independant (due to external effects).
- deterioration is a continuous, non-regressive process, which inevitably leads to an end-of-life state.

The aforementioned assumptions determine the design of the Markov chain to use:

- Because of the non-regressive character of the deterioration process, the Markov chain has to be non-intercommunicating.
- Furthermore, the end-of-life state should be absorbing, that means $p_{nn} = 1$.
- In order to decouple the age-dependant and age-independant failures, it is possible to asign a failure rate to each deterioration state. This approach differs from other proposed models [1, 2, 3] so far, that a failure is considered as an event which may occur while residing in one state, rather than a separate fault-state.
- As the deterioration is a staigtforward process there should exist only one passage through each of the finite states.

Figure 1 schematically shows a Markov chain which meets the requirements above.

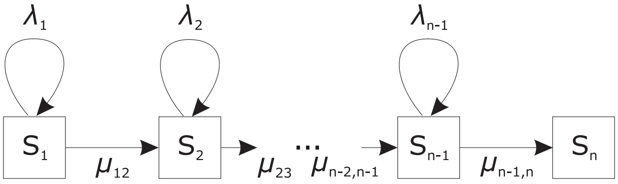

Figure 1: Markov chain for the representation of failure distributions

It is obvious that the Markov chain of figure 1 is non-ergodic and therefore no stationary distribution exists. Morever it can be seen that the failure rates λ_i associated to state S_i comprise of an age-independant part, which is equal for all states, and of an age-dependant part. State S_n (the end-of-life state of the component) is absorbing and does not posses a failure rate, as retired equipment cannot fail any more.

3 IMPLEMENTATION

In this section case-studies are presented conserning the reproduction of failure distributions by means of Markov chains. As a first step, a fictitious typical bathtub distribution was approximated both through Weibull distribution and the Markov chain of figure 1. In a second step a real failure distribution of high-voltage SF_6 circuit breakers over the operating time was approximated through the Markov chain.

3.1 *Approximation of a typical bathtub distribution*

The approximation of failure distributions over the operating time was applied for a typical bathtub curve. Bathtub curves are commonly used for the representation of failure distributions of mechanical and electrical components featuring wear, mainly because of their possibility to modell both the teething and ageing effects. Mathematically, bathtub failure curves are derived from Weibull distributed failure rates. Figure 2 shows the failure distribution curve which was modelled by means of a time-homogeneous Markov chain, as introduced in section 2.

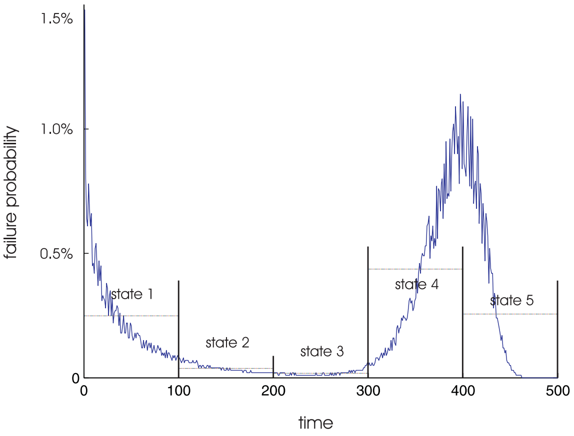

Figure 2: Approximation of Weibull distributed failure probabilities

For the modelling by the Markov chain following procedure was followed:

- Determination of the model size: at first the number of the discrete states is set.

- Determination of each state's residence time $t_{res,i}$: the most apparent solution is to divide the time axis in equal intervals.

- Calculation of the transition rates: because of the trivial design of the Markov chain, the transition rates of each state equals the reciprocal of its residence time $\left(\mu_{i \to i+1} = t_{res,i}^{-1}\right)$.

- Calculation of the mean failure probability for each state: with the assumption that all units in one state are uniformly distributed, the mean failure probability is the simple average of all years' failure probabilities for the specific state.

In figure 2 the aforementioned procedure is schematically shown for the case of five discrete states (the end-of-life state is not included, as it features an infinite residence time). The dotted line is the mean failure probability for each of the five states. The assumption of uniform distribution within each state mentioned before forms a contraint for the representation through time-homogeneous Markov chains. Moreover, the size of the Markov chain would seem to have an influence on the accuracy of the representation.

In order to investigate the impact of these two parameters (model size, uniformity of the distribution) on the accuracy of the representation, following simulation was performed: on the basis of figure's 2 failure probability distribution, Markov chains of different size were configured. Objective was the simulation of the failure distribution over time for a given fleet of units (figure 3).

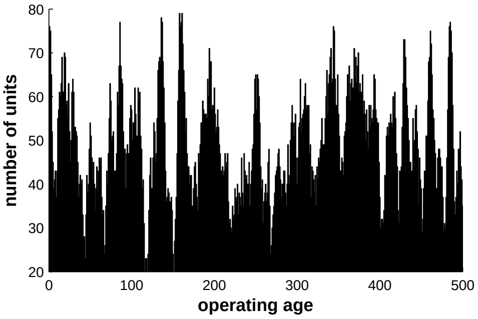

Figure 3: Age distribution of the modelled fleet

In order to account for variable uniformity of the age distribution of the fleet, a modification of the original distribution of figure 3 was performed, where the original age distribution a_n was biased around the mean value $\overline{a}$ with the factor j/k. The modified age distribution is then given by:

$$b_n^{(j)} = a_n - \frac{j}{k} \cdot (a_n - \overline{a}) \qquad j \in 1, \ldots, k$$

$$n \in 1, \ldots, N$$

where the mean value $\overline{a}$ can be calculated by:

$$\overline{a} = \frac{1}{N} \cdot \sum_{n=1}^{N} a_n \qquad n \in 1, \ldots, N$$

A schematical representation for the modification of the original age distribution is given in figure 4 (abstract) for three degrees of uniformity.

The simulation results are given in figure 5, where the computational error of the Markov chain is plotted against the size of the model and the deviation of the initial age distribution from the uniform distribution. In figure 5 all values are normalised in relation to their maxima, as resulting from the simulation. For the assessment of the computational error, the failure distribution was modelled (besides the simulation with the time-homogeneous Markov chain) by an exact model, assuming that the failure probability is known for each operating age, and the results of the two appoaches were compared. The calculation of the aforementioned deviation was made according to:

$$\sigma = \sqrt{\frac{1}{N} \cdot \sum_{n=1}^{N} (a_n - \overline{a})^2} \qquad (2)$$

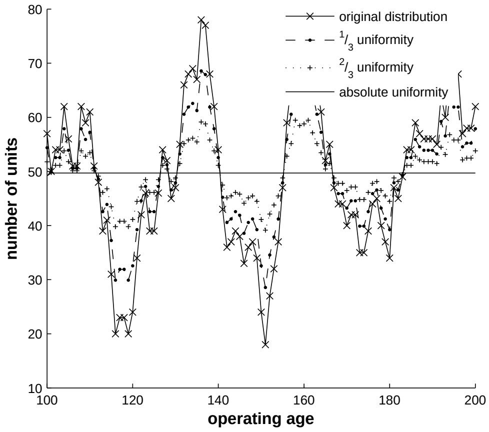

Figure 4: Modification of the original age distribution towards a uniform distribution

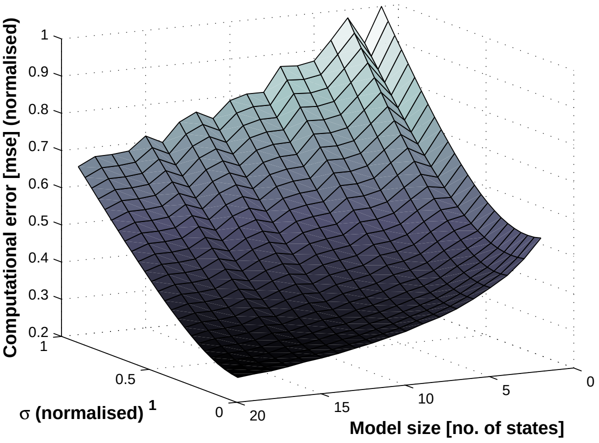

Figure 5: Computational error vs. size of Markov chain and uniformity of the fleet's age distribution (^{1} deviation calculated according to equation 2)

As may be seen from figure 5 the computational error of the time-homogeneous Markov chain representation rises with increasing nonuniformity of the initial age distribution and decreases with increasing model size. The reason for this behaviour pattern bears on the approximation of the (non-constant) failure probabilities within each state by their (constant) simple average. The simple average is a good approximation only as long as the units to be modelled are nearly uniformly distributed in each state. The more the distribution of the units inside one state differs from the uniform distribution the less accurate will the approximation by means of the simple average be. Moreover, the higher the number of the different states, the better will the approximation for a given age distribution be, as may be intuitionally be seen from figure 2.

3.2 *Approximation of a real SF_6 circuit breakers failure distribution*

As a second step, a real failure distribution of SF_6 high-voltage circuit breakers was modelled by a Markov chain. The original data are taken from parts of the German high-voltage power grid and capture more than 6500 circuit breakers. The failure frequency of the investigated circuit breakers is given in figure 6.

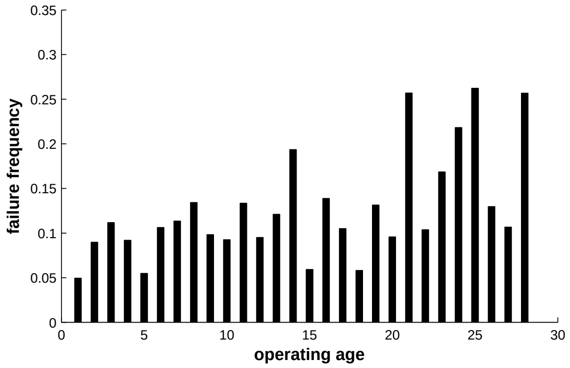

Figure 6: Failure frequency of the modelled SF_6 circuit breakers

The failure frequency does not follow a Weibull distribution and also displays a quite high degree of randomness. However, the time-homogeneous Markov chain representation is rather immune to randomness, as it uses the mean failure frequency for each state. A Markov chain with four states (brand new, rather new, rather old, old) was configured and afterwards tested on a real fleet of circuit breakers. The age distribution of the units is given in figure 7.

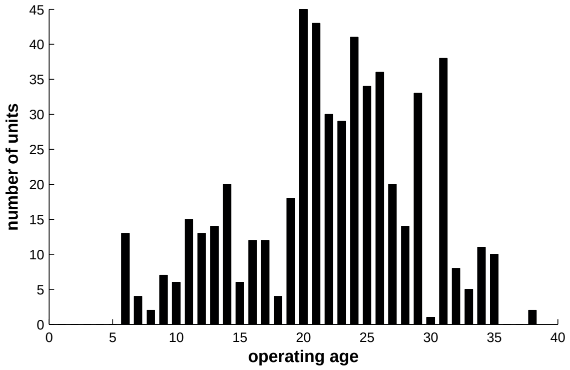

Figure 7: Age distribution of the test set

It is obvious from figures 6 and 7 that no reliable failure information exist for the older equipment. That would mean that for all units with an operating age more than 28 years no failure would be simulated if the last state (old) stretched only over operating ages higher than 28. In order to solve this problem three approaches are possible:

- to force the older units to retire at the maximum operating age, for which failure data are available.
- to choose the states in such a manner, that each state is defined by sufficient failure information (adaptive Markov chain). However, this may cause a deformation of the asymptotical behaviour of the Markov chain, as will be seen.
- to extrapolate the failure frequency for the older equipment. The extrapolation is, however, a difficult task, mainly because of the randomness and the lack of knowledge about the complex deterioration process.

All three approaches were simulated and compared to the ”exact” solution, which would be given if the failure frequency for each operational age was known and the units were forced to retirement after the maximum operating time, for which failure data are available. The results of the simulation are given in figure 8 for Markov chains each with four states.

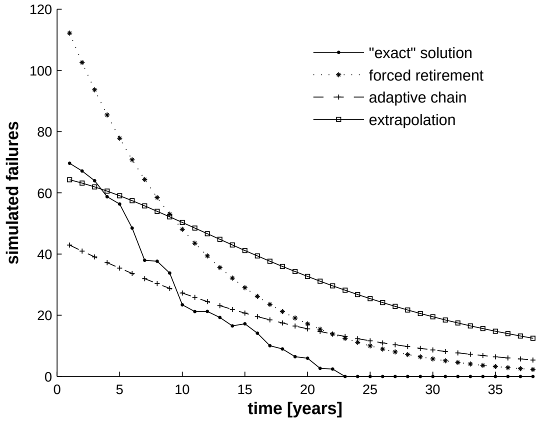

Figure 8: Simulation results for the SF_6 circuit breakers fleet

As can be seen, all approaches lie above the ”exactly” simulated results, which means that the approximation of failure distributions by means of time-homogeneous Markov chains is rather conservative. In the present example, the approximation error is mainly due to the missing failure information about older units, which were merely not taken into account in the ”exact” simulation.

However, on the basis of the two examples presented so far, some conclusions may be drawn:

- Time-homogeneous Markov chains of the figure 1 type are -in general- incapable of reproducing various distributions. Their capabilities of reproducing various distributions is directly proportional to their size.
- For Markov chains with few states, the main problem of the approximation lies in their asymptotical behaviour, where, due to the constant transition rates, an inverse exponential curve is generated. Eventually, the approximation by means of small Markov chains becomes an adaptation task, where such inverse exponential curves have to be fitted to the given distribution.

4 IMPROVEMENTS

As described previously, time-homogeneous Markov chains introduce an approximation error when used to reproduce failure distributions due to ageing. In this section some possible adaptation procedures are proposed for the configuration of the Markov chains.

In the examples given in section 3 the residence time of the chain's states was determined by simpy dividing the time axis in equal intervals. However, the failure distribution to be modelled may contain statistical information valuable for the chain's configuration. Such information can be extracted by means of decomposition methods [4]. Decomposition algorithms normally consider a time series X_t as composed of following components: season (S_t), randomness (R_t), trend (T_t) and cycle (C_t). Equation 3 defines a multiplicative decomposition model:

$$X_t = T_t \cdot S_t \cdot C_t \cdot R_t \tag{3}$$

For the determination of the states' residence times, only components with a periodic behaviour are of interest, namely the season and the cycle. It may be plausible that an adaptation of the chain's residence times to the cycle period of these components leads to better representation.

On the basis of the example given in section 3.2 a decomposition algorithm was executed, leading to following conclusions:

- as the scale of the input time series are years, no seasonal component S_t exists,
- there is a small positive trend T,
- the random component R_t is quite dominant, and
- a certain cycle could be observed and segregated.

The number of failures over time of section 3.2 was simulated again by using a Markov chain of the same size (four states), however, with residence times adapted to the cycle period of the failure distribution given in figure 6. A marginal improvement was apparent, the comparison of the two Markov chain approximations is given in figure 9 (Markov chain I: equal residence time for all states, Markov chain II: states adapted to the cycle of the failure distribution).

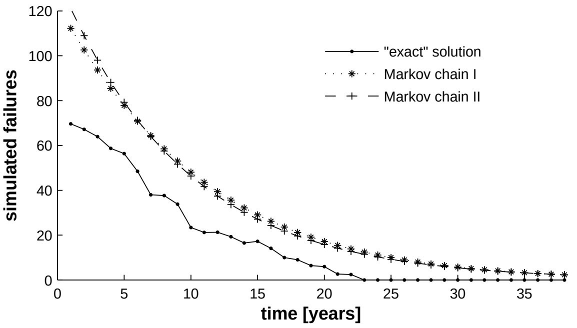

Figure 9: Comparison of uniform-state and adaptive Markov chains at the SF$_6$ circuit breakers example

Besides the utilisation of information about the cycle of the failure distribution, another possible improvement of the time-homogeneous Markov chain representation could be to introduce a correction factor, in order to smooth out the rather conservative forecast. This correction factor could be applied both for the adjustment of the states' residence times and the states' failure rates. Figure 10 shows an example application of two possible correction factors for the example in section 3.2.

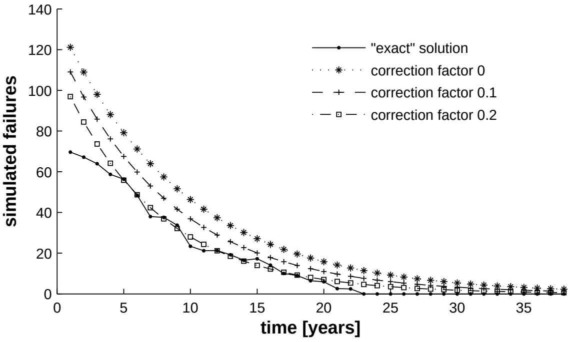

Figure 10: Application of correction factors at the SF$_6$ circuit breakers example

Another improvement possibility is the use of time-variant Markov chains for the approximation of failure distributions. Time-homogeneous Markov models are known for their calculational simplicity and this is also the reason why they were exclusively used in this paper. However, there exist also time-variant Markov chains which feature almost the same calculational simplicity. One such model is proposed in [5], where a Markov model is introduced, featuring transition rates which follow a Weibull distribution. The Weibull-Markov stochastic model can demostrate higher representational capabilities than a time-homogeneous Markov chain, without a significant rise in required computational resources.

5 CONCLUSIONS

In the present paper the approximation of failure distributions due to ageing by means of time-homogeneous Markov chains was investigated. The decision to use time-homogeneous models was made on the one hand because of the computational simplicity and on the other hand because of the possibility to integrate such a model into a more general asset management tool featuring maintenance, replacement and costs investigations. The design requirements lead to a non-ergodic, straigthforward Markov chain with the failure modelled as an event rather than a seperate state. Instead, the chain features an absorbing state representing the equipment's end-of-life.

After introducing two exemplary calculations, conclusions were drawn and some drawbacks were identified. The main disadvantage of time-homogeneous Markov chains seems to be that the constant transition rates invokes an inverse exponential asymptotical behaviour, which in particular for small-sized models, makes the models incapable to represent various distributions.

Some improvement proposals were listed in section 4, thereunder the adaptation of the Markov models to the cycle characteristics of the distribution to be modelled, the use of a correction factor in order to smooth the rather conservative forecast of the Markov representation, as well as the possible use of time-variant Markov models.

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APPENDIX

Definition A.1 (finite Markov chain). A finite Markov chain is a sequence of random variables $X_1, X_2, \ldots, X_n \in S$, which satisfy the Markov property. The countable set S is the state space of the chain.

$$\begin{aligned}&\Pr(X_{n+1} = x | X_n = x_n, \ldots, X_1 = x_1)\\&\quad = \Pr(X_{n+1} = x | X_n = x_n)\end{aligned} \tag{A.1}$$

Similar to the completely memoryless Markov chain of definition A.1, Markov chains with a memory of m incidents can be defined.

Definition A.2. A m-order Markov chain is a sequence of random variables $X_1, X_2, \ldots, X_n \in S$, which satisfies following property:

$$\begin{aligned}&\Pr(X_n = x_n | X_{n-1} = x_{n-1}, \ldots, X_1 = x_1)\\&\quad = \Pr(X_n = x_n | X_{n-1} = x_{n-1}, \ldots, X_{n-m} = x_{n-m})\end{aligned} \tag{A.2}$$

Definition A.3 (transition matrix). A finite Markov chain X_t over the state space S may be described by a stochastic matrix **P** (called transition matrix) with following properties:

$$\begin{aligned}&p_{ij} = \Pr(X_{n+1} = j \mid X_n = i)\\&p_{ij} \geq 0 \quad \forall \quad i, j \in S\\&\sum_{j \epsilon S} p_{ij} = 1\end{aligned} \tag{A.3}$$

p_{ij} is the transition probability from state i to state j in one step.

If the transition probabilities p_{ij} are constant over time, then the (finite) Markov chain is *time-homogeneous*. In that case the transition probabilities are given by $\mathbf{P}^k$. This is a direct conclusion from the Chapman-Kolmogorov theorem.

Definition A.4 (stationary distribution). A distribition over the state space S is called stationary if it satisfies the following property:

$$\pi = \pi \cdot \mathbf{P} \tag{A.4}$$