

# AN APPROACH TO EVALUATING ACCURACY OF NUMERICAL SIMULATION OF LINEAR NETWORK TRANSIENTS

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**Abstract** – This paper discusses an approach to evaluating accuracy of numerical integration in simulation of linear electrical network transients. Both modes of the free motion of a linear network (i.e. modes of the general solution of the differential equations) and modes of the forced motion (particular solution) are distorted independently when numerical integration is performed. The distortions of the modes are considered as accuracy characteristics of the method applied. For the free motion modes the quantities used are damping and frequency distortions, while for the forced motion modes of sinusoidal form the quantities used are amplitude and phase distortions. Each distortion is defined as the relative error when reproducing the corresponding parameter. Discussed are cases of extreme distortions that determine principal numerical distorting mechanisms in simulations of network transients by Runge-Kutta and multi-step methods. Evaluating the distortions of the free motion modes brings understanding of the accuracy and stability of a method and simulations, unachievable when using truncation error estimations.

**Keywords:** *linear network, transient, numerical integration, Runge-Kutta method, multi-step method, accuracy, stability*

## 1 INTRODUCTION

Numerical simulation of transients in linear electrical networks is based on integration of the set of its ordinary differential equations (ODEs). In case of using discrete companion models [1] a set of algebraic equations is solved instead of directly integrating the ODEs, however, the simulation results are similar to what would be obtained using the integration method to which the companion model is associated.

The accuracy characteristics of many integration methods (algorithms) applied to solve ODEs for specific engineering problems are known in terms of truncation and global error [1-3]. However, these quantities are far from being comprehensive accuracy characteristics of the methods and often are not sufficient for understanding numerical phenomena observed in engineering simulations.

This paper discusses an approach to evaluating accuracy of integration, based on using distortions of the solution modes. For the free motion modes the quantities used are damping and frequency distortion, while for the forced motion modes of sinusoidal form the quantities used are amplitude and phase distortion. Each distortion

is defined as the relative error when reproducing corresponding parameter.

The general idea of the approach is a comparison between the true solution of a set of ODEs and the solution of difference equations, approximating the ODEs when numerical integration is performed. The basics of this approach with respect to reproduction of the free motion and for the simplest explicit Runge-Kutta methods were developed in [4] when the problem of modeling networks with valve converter bridges was studied. The definition given in [4] for the distortion of damping was not optimal, which did not allow revealing important method properties. Then there have been published some other papers where quantities, pretty similar to mode distortions, were introduced (examples can be found in [5,6]). However, in [5,6] effects of distorting damping and frequency were not considered independently.

Later in [7] more reasonable distortion definitions were given and new accuracy characteristics of the basic explicit methods were obtained. The revised approach was applied to other methods and algorithms (implicit Runge-Kutta methods [7], including the Pade sets [7-9], linear multi-step methods [10], modifications of conventional methods [7-9], the Richardson extrapolation [11], combined algorithms [12], etc.). Also, a similar idea was used to evaluate accuracy for the forced motion of a network under sinusoidal applied forces [13].

In this paper the distortion-based approach is discussed and generalized. It is shown that mode distortions as accuracy characteristics bring understanding of the properties of a method as well as of numerical phenomena observed, unachievable when using truncation error estimations.

## 2 MATHEMATICAL MODEL OF A LINEAR NETWORK AND NUMERICAL METHODS CONSIDERED

Let us describe a linear network to be simulated with a set of ordinary differential equations of the form

$$\frac{d}{dt}\mathbf{x}(t)=\mathbf{A}\mathbf{x}(t)+\boldsymbol{\varphi}(t), \mathbf{x}(0)=\mathbf{x}_0, \quad (1)$$

where  $\mathbf{x}(t)$  is a column of  $n$  state variables,  $\mathbf{A}$  is an  $n*n$  constant matrix,  $\boldsymbol{\varphi}(t)$  is a column of  $n$  applied forces,  $t$  is time.

### 2.1 Runge-Kutta Methods

Numerical integration of ODEs (1) by a single-step  $r$ -stage Runge-Kutta method is described by

$$\mathbf{x}_{k+1} = \mathbf{x}_k + h \sum_{i=1}^r g_i y_i, \quad (2a)$$

$$y_i = \mathbf{f}(t_k + hc_i, \mathbf{x}_k + h \sum_{j=1}^r d_{ij} y_j), \quad (2b)$$

where  $h$  is the time step size,  $k$  is the number of a step,  $g_i$ ,  $c_i$ , and  $d_{ij}$  are coefficients specific for each method, defined by the Butcher Table [3]. Further  $h$  is assumed to be constant.

When the method is applied to the homogeneous set of ODEs, the stability of numerical integration is characterized by the stability function  $F(z)$  which can be calculated using the parameters of equations (2), including constants of the Butcher Table. The difference equation describing the numerical integration has the form

$$\mathbf{x}_{k+1} = F(Ah) \mathbf{x}_k. \quad (3)$$

### 2.2 Linear Multi-Step Methods

Numerical integration of ODEs (1) by a linear  $m$ -step (single-stage) method is described by

$$\mathbf{x}_{k+1} = \sum_{i=0}^{m-1} a_i \mathbf{x}_{k-i} + h \sum_{i=-1}^{m-1} b_i \mathbf{f}(t_{k-i}, \mathbf{x}_{k-i}), \quad (4)$$

where  $a_i$  and  $b_i$  are coefficients. For the homogeneous set of ODEs the corresponding difference equation is

$$\mathbf{x}_{k+1} = (\mathbf{I} - b_{-1}Ah)^{-1} \cdot \sum_{i=0}^{m-1} (a_i \mathbf{I} + b_i Ah) \mathbf{x}_{k-i}. \quad (5)$$

## 3 THE BASICS OF THE APPROACH

The accuracy of a method and the accuracy of numerical simulations of the network transients are closely connected, but separate issues. Therefore analysis of the accuracy of a method in terms of distorting parameters, describing the form of the simulated transient, seems to be promising. It is assumed that properties of the simulated linear network are defined by the spectrum of matrix  $\mathbf{A}$ .

For a linear network the ODEs (1) are linear and therefore all the modes of the free motion (general solution) and of the forced motion (particular solution) are distorted independently when numerical integration is performed. To evaluate the accuracy of both the simulation and the method applied, distortions of damping and frequency can be used for the free motion modes, while amplitude and phase distortions can be used for the forced motion modes. Each distortion is defined as the relative error when reproducing the corresponding pa-

rameter. A simulation is considered to be accurate if the important solution modes are reproduced satisfactory, while other modes are suppressed or at least do not cause numerical problems.

### 3.1 Reproduction of the Free Motion Modes

The free motion is determined by the homogeneous part of ODEs (1) and contains  $n$  modes  $\xi(t)$ , corresponding to the eigenvalues  $\lambda = \alpha + j\omega$  of matrix  $\mathbf{A}$ . Each of  $n$  modes  $\xi(t)$  is characterized by its damping  $\eta$  and frequency  $\omega$ . In the scope of the approach discussed here it is reasonable to define damping  $\eta$  as the ratio of the upper envelope of  $\xi(t)$  at  $t=0$  to its value at  $t=h$ .

Numerical integration transforms each original mode  $\xi(t)$  into a numerical mode (for the single step method) or into  $m$  numerical modes (for the  $m$ -step method). The damping  $\eta_r(h)$  and frequency  $\omega_r(h)$  of each numerical mode are defined in the same way as for the original mode. Comparison between corresponding original and numerical modes in terms of their damping and frequency distortions, giving the understanding of numerical phenomena observed, can be used for evaluating accuracy of the method.

### 3.2 Reproduction of the Forced Motion Modes

To analyze reproduction of the forced motion, it is reasonable to consider a scalar (testing) ODE of the form

$$\frac{d}{dt} x(t) = \lambda x + \varphi(t). \quad (6)$$

where  $\varphi(t)$  is an applied force,  $\lambda$  is a complex number. In practical simulations of linear networks the case of a sinusoidal applied force  $\varphi(t)$  of the frequency  $\beta$  is the most interesting. Numerical integration transforms the original forced motion with a complex amplitude  $a$  into a numerical motion of sinusoidal form of the same frequency  $\beta$ , but with a different complex amplitude  $a_r$ . If the applied force  $\varphi(t)$  is a sum of harmonics of different frequencies, each of the harmonics causes a distorted harmonic mode of the same  $\beta$  in the numerical solution. Comparison between corresponding forced motion modes in the true and numerical solutions in terms of amplitude and phase of their complex amplitudes  $a$  and  $a_r$  can be used for evaluating accuracy of reproduction of the forced motion.

## 4 DISTORTIONS OF THE FREE MOTION MODES

Let us define distortions of the damping and frequency as

$$\delta_\eta = \eta_r(h)/\eta - 1, \quad \delta_\omega = \omega_r(h)/\omega - 1. \quad (7)$$

When a Runge-Kutta method (2) is applied, for each original mode  $\xi(t)$ , corresponding to the eigenvalue  $\lambda$ , the distortions are calculated as

$$\delta_\eta = \frac{\exp(\operatorname{Re}z)}{|F(z)|} - 1, \quad \delta_\omega = \left| \frac{\arg F(z)}{\operatorname{Im} z} \right| - 1, \quad (8)$$

where  $z = \lambda h$ .

For a linear  $m$ -step method (4) the difference equation (5) gives a characteristic equation of the form

$$z_r^m = (1 - b_{-1}z)^{-1} \sum_{i=0}^{m-1} (a_i + b_i z) z_r^{m-1-i}. \quad (9)$$

with  $m$  roots  $z_r$  ( $i=1,2,\dots,m$ ), therefore each original mode  $\xi(t)$  causes  $m$  numerical modes. The distortions for a numerical mode are calculated using formulas (8), but, instead of  $F(z)$ , the corresponding root of the difference equation (9) should be substituted. More detailed derivations of distortion formulas for the free motion modes are given in [7,10].

Let the free motion of the simulated network contain a mode  $\xi(t)$  that corresponds to eigenvalue  $\lambda$ . If a method, either single-step or multi-step, is applied in simulation with constant time step size  $h = \text{const}$ , the distortions calculated according to formulas (8) characterize the accuracy of reproduction of the mode  $\xi(t)$  in the numerical solution.

## 5 DISTORTIONS OF THE FORCED MOTION MODES

Let us define distortions of the amplitude  $A$  and phase  $\psi$  as

$$\delta_A = \frac{A_r}{A} - 1 = \left| \frac{a_r}{a} \right| - 1, \quad (10a)$$

$$\delta_\psi = \frac{\psi_r}{\psi} - 1 = \frac{\arg a_r}{\arg a} - 1, \quad (10b)$$

where  $a$  and  $a_r$  are the complex amplitudes of the original and numerical mode, respectively. To calculate the distortions for a harmonic mode, one should know both the complex amplitudes. Finding the original amplitude is trivial. Let us consider the amplitude of the corresponding numerical mode, assuming  $\varphi(t) = \exp(j\beta t)$ .

When a Runge-Kutta method (2) is applied to the test equation (6) with an applied force  $\varphi(t) = \exp(j\beta t)$ ,

$$x_{k+1} = F(z)x_k + h \sum_{i=1}^r B_i(z) f(t_k + hc_i), \quad (11)$$

and the numerical mode complex amplitude  $a_r$  is

$$a_r = \frac{\sum_{i=1}^r B_i(z) \exp(jc_i s)}{\exp(js) - F(z)} h, \quad (12)$$

where the functions  $B_i(z)$  are calculated based on formulas from [13],  $s = \beta h$ .

When an  $m$ -step method (4) is applied to the test equation (6) with the applied force  $\varphi(t) = \exp(j\beta t)$ , the complex amplitude  $a_r$  is

$$a_r = \frac{\sum_{i=-1}^{m-1} b_i \exp(-jis)}{\exp(js) - \sum_{i=0}^{m-1} a_i \exp(-jis) - z \sum_{i=-1}^{m-1} b_i \exp(-jis)} h. \quad (13)$$

More detailed derivations of distortion formulas for free motion modes are given in [13].

In actual simulations the selected value of  $h$  should provide appropriate accuracy of reproduction of both the free and forced motions of the network. If  $\beta$  is far greater than, for example, the frequencies of important free motion modes (supposed to be reproduced accurately), care should be taken of amplitude and phase distortions of the forced motion.

However, usually  $\beta$  is far less than the highest frequencies of important free modes and/or than the maximum inverted time constants  $1/\alpha$  of the network. If  $h$  is selected with respect to reproduction of the free motion modes, it also provides acceptable or even high accuracy for the forced motion. This takes place because amplitude and phase distortions (10) turn to be very small. For example, the Euler method (explicit) and backward Euler method (implicit) are known to have relatively poor accuracy in terms of truncation error. However, for both the methods values of  $|\delta_A|$  and  $|\delta_\psi|$  are about 0.01 for  $z = -0.5$ ,  $s = 0.1$  and are less than 0.001 for  $z = j0.5$ ,  $s = 0.1$ . Therefore further we will focus on the accuracy of the free motion modes only.

## 6 EXAMPLE DISTORTION CHARTS FOR THE FREE MOTION MODES

Mapping lines of constant damping and frequency distortions  $\delta_\eta = \text{const}$  and  $\delta_\omega = \text{const}$  in the complex plane  $z$  (with axes  $\alpha h$ ,  $j\omega h$ ) gives universal accuracy characteristics of a method, containing essential, *a priori* information on numerical solution properties. Such a plot built for a method in the upper half-plane will be referred to as its distortion chart. In the distortion chart it is convenient to show also the known stability borderline of the method.

Another useful form of presenting accuracy characteristics is plotting the distortion curves along the axes  $\alpha h$ ,  $j\omega h$ . These curves are convenient for comparing methods to find out which of them will be more appropriate in simulations of a specific network. The network properties are obviously defined by the spectrum of its matrix  $A$ .

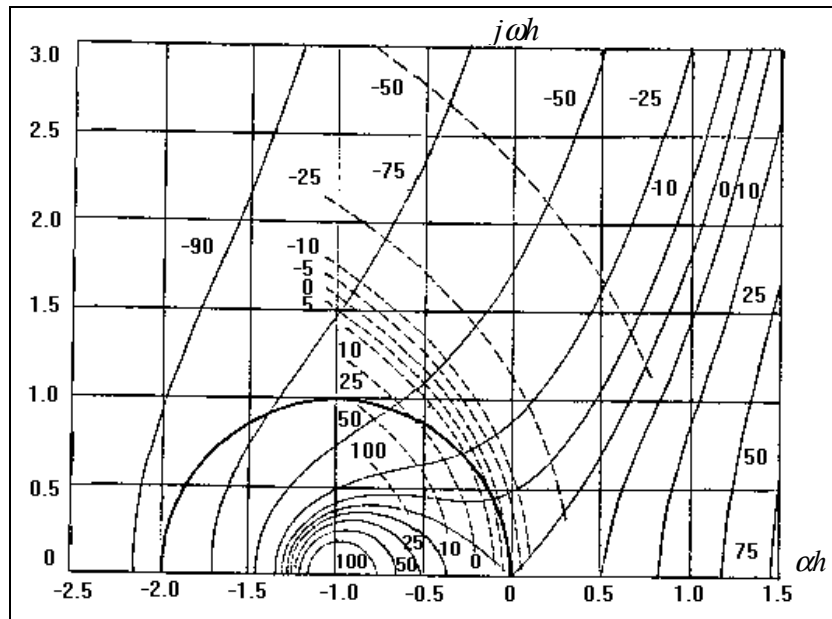


Figure 1: Distortion chart for the Euler method

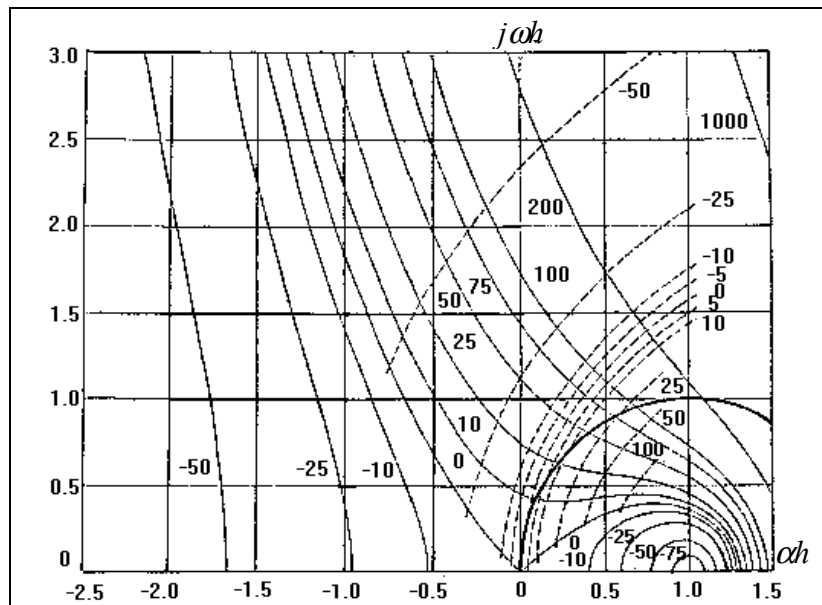


Figure 2: Distortion chart for the backward Euler method

### 6.1 Example Charts for Runge-Kutta Methods

The Euler (explicit) and backward Euler (implicit) methods are single-step, single-stage methods with stability functions  $F(z)=1+z$  and  $F(z)=1/(1-z)$ , respectively. Their distortion charts are shown in Fig. 1 and 2 (solid lines for distortion of damping, dotted lines for distortions of frequency, the values are given in percent, the boldface line is the known stability borderline).

In the chart of Fig. 1 attention should be paid to the area around  $z = -1$  with concentrated lines  $\delta_\eta > 0$ . These lines concentrate around the point  $z = -1$  corresponding

to the root of the stability function  $F(z)$  ("method zero point"). If a simulated transient contains a mode  $\xi(t)$  with  $z = \lambda h$  close to  $z = -1$ , the numerical mode caused by  $\xi(t)$  quickly disappears in the numerical solution. For an explicit  $r$ -stage method ( $r > 1$ ) the number of zero points is  $r$ , and the distortion lines topology becomes more complicated. In some vicinity of a method zero point distortions  $\delta_\eta$  and  $\delta_\omega$  change considerably, therefore even a slight change in the value of  $h$  (that causes the point  $z = \lambda h$  move within this vicinity) may result in dramatical change in reproduction of the corresponding mode  $\xi(t)$ . This effect is especially important for multi-

stage explicit methods whose zero points with the maximum real part is very close to the axis  $j\omega h$  (for example, for the standard 4<sup>th</sup> order Runge-Kutta method). The capability of suppressing modes in numerical solution [7-9] and filtering properties of some methods considered in [15] are closely related to this effect.

### 6.2 Example Charts for a Multi-Step Method

The Adams-Bashworth 2 method is an explicit two-step method of the form

$$x_{k+1} = x_k + \frac{h}{2} [3f(t_k, x_k) - f(t_{k-1}, x_{k-1})] \quad (14)$$

Its characteristic equation

$$z_r^2 = (1 + \frac{3}{2}z)z_r - \frac{1}{2}z \quad (15)$$

has two roots ( $i=1,2$ ):

$$z_r^{(i)} = u + (-1)^{i+1} \sqrt{v} \quad (16)$$

$$u = \frac{1}{2} + \frac{3}{4}z, \quad v = \frac{1}{4} + \frac{1}{4}z + \frac{9}{16}z^2. \quad (17a,b)$$

Roots (16) define the two numerical modes caused by each  $\lambda = z/h$ . Therefore, two different distortion charts, each of them for one of the roots, are needed. However, more convenient is another pair of charts: the first chart corresponds to the root with greater modulus, the second chart corresponds to the other root (see Figures 3 and 4, respectively).

The character of distortions is different for both the roots. In a small vicinity of  $z=0$  one of numerical modes (the "principal" root) accurately reproduces the original mode  $\xi(t)$  that corresponds to  $\lambda=z/h$  (Fig. 3). The other numerical mode (Fig. 4) has a very large damping distortion  $\delta_\eta$ , which results in this mode quickly disappearing in the numerical solution (a "parasite" root). The farther the point  $z=\lambda h$  is from  $z=0$ , the more different is the character of the distortions for both the numerical modes. The relationship between the distortions and, consequently, between both the numerical modes changes depending on the direction in which the point  $z$  moves in the complex plane. Specifically, for  $z=(-2+j4\sqrt{2})/9$  the two numerical roots have the same value; if one of the values of  $z$  for a simulated network is close to this point, there may appear an illusion of two clearly distorted numerical modes caused by one original.

In a more general case of an  $m$ -step method (4) with  $m$  roots of equation (9) there are  $m$  pairs of distortions (8), and  $m$  distortion charts are needed. The greater  $m$  is, the more complex the distortion chart layout becomes. In different parts of the complex plane the relationship between the roots  $z_r$  ( $i=1,2,\dots,m$ ) change substantially, and the situation when one of the roots is "principal"

while other  $m-1$  roots are "parasite" does not always take place.

## 7 EXTREME DISTORTIONS AS REASONS OF PRINCIPAL NUMERICAL MECHANISMS

Both distortions  $\delta_\eta$  and  $\delta_\omega$  change in the interval  $[-1, \infty)$ . If  $\delta_\eta = \infty$ , the numerical mode disappears at the very first time step  $k=1$ ; if  $\delta_\eta = -1$ , at  $k=1$  the mode becomes infinity (utmost numerical instability). Two other cases of extreme distortions,  $\delta_\omega = \infty$  and  $\delta_\omega = -1$ , determine other principal numerical mechanisms.

One of these mechanisms is observed when a quickly damping mode, either aperiodical ( $\omega=0$ ) or low frequency oscillatory, is reproduced. Such a mode can be transformed into a poorly damped, pseudo-oscillatory numerical mode with illusion of high frequency  $\omega(h)=\pi/h$ . This occurs if  $\delta_\omega = \infty$  or it is very large. The trapezoidal rule (method Pade-1,1) is known to be apt to produce numerical oscillations when  $h$  is relatively large, which is a result of  $\delta_\omega$  being close to infinity. However, this effect takes place when using other Pade methods also. Specifically, for a single-step Pade- $r,r$  method with an odd number of stages  $r$  the stability function  $F(z)$  tends to  $-1$  as  $\omega h$  approaches  $\infty$ , therefore any method of this kind is apt to produce high frequency numerical oscillations. Furthermore, for any explicit single-step method whose stability function has at least one negative real root such oscillations can appear [9].

Another principal distortion mechanism is observed when a poorly damped, high frequency mode is reproduced. Such a mode can be transformed into a poorly damped, low frequency numerical mode. This occurs if  $\delta_\eta = 0$  or it is very small, while  $\delta_\omega$  is close to the lower limit ( $-1$ ). For example, any "symmetrical" Pade- $r,r$  method (for both even and odd  $r$ ) provides zero damping distortion ( $\delta_\eta = 0$ ) for any pure oscillatory mode  $\omega h = 0$  (this is because the polynomials of the numerator and denominator of  $F(z)$  are symmetrical), while the frequency distortion  $\delta_\omega$  tends to  $-1$  for large  $\omega h$ . Specifically, it takes place for the trapezoidal rule. All Pade- $r,r$  ("symmetrical") methods are known to be A-stable, which allows using large  $h$ . However, the larger  $h$  is, the greater the total distortion of a high frequency mode can be. Distortions of this kind result in transforming high frequency sustained oscillations of physical nature into low frequency sustained phantom oscillations.

Usually, but not always, both the numerical mechanisms described here cause numerical problems in actual network simulations when  $h$  is large. Therefore, these mechanisms seldom affect the solution considerably when using explicit methods, either single-step or multi-step, because of their relatively small stability regions. However, these effects are capable of substantially distorting numerical solution when using A-stable implicit methods with no stability limitations for  $h$ .

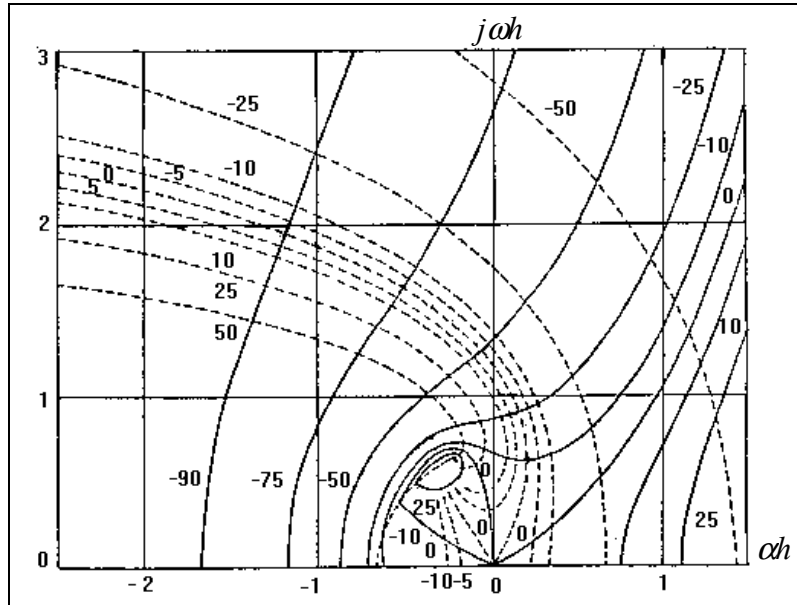


Figure 3: Distortion chart for the Adams-Bashfort 2 method, the root with greater modulus

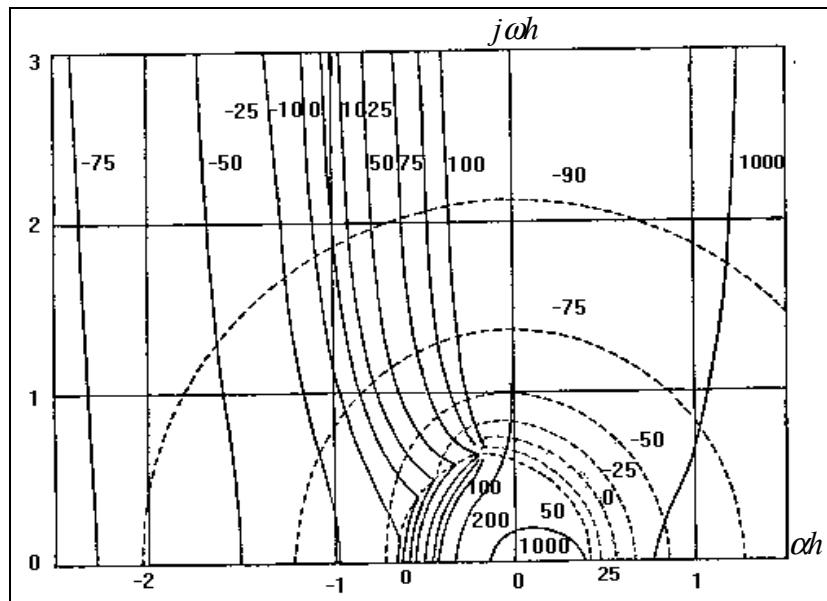


Figure 4: Distortion chart for the Adams-Bashfort 2 method, the root with smaller modulus

## 8 DISCUSSING THE APPROACH

The accepted concept of accumulating truncation error from step to step as a reason for divergence between the numerical and true solutions usually results in a conclusion that the longer the simulation interval is, the greater this difference can be. However, whereas the global difference between the true solution of ODEs (1) and their numerical solution increases with elongating the simulation interval, distortions of the modes are constant. It is also worth noticing that numerical instability phenomena cannot be understood in terms of truncation error and their analysis has always been based on other analytical tools. The distortion-based approach allows connecting the concepts of numerical

accuracy and stability as having common nature (distortion of damping). This approach describes accuracy in “electrical” terms (damping, frequency, etc.), which is convenient in electrical engineering applications.

A simple regular procedure allows comprehensively describing the accuracy of any single-step (Runge-Kutta) or linear multi-step method. Such analysis has been performed for numerous methods and algorithms, both single-step and multi-step, both explicit and implicit [7-10, 13, 14]. The approach was also applied in studying the Richardson extrapolation [11] and cyclic combined algorithms [12]. Although originally developed for the linear networks (with linear ODEs (1)), the approach is applicable to some non-linear problems. Specifically, it has been used to reveal and analyze pseudo-stochastic processes in simulations of non-

linear network [15] and considerable voltage distortions in thyristor converter simulations.

The results obtained show that for each mode  $\xi(t)$  the functions  $\delta_\eta(h)$  and  $\delta_\omega(h)$  are always non-linear and often non-monotonous. Usually a method with lower truncation error provides weaker distortions, and these distortions reduce with decreasing  $h$ , which is hardly surprising. However, in specific areas of the complex plane  $z$  methods with the same truncation error can provide dramatically different distortions, which results in substantial difference in reproducing modes  $\xi(t)$  whose  $z=\lambda h$  belongs to those specific areas. On the other hand, a method with a worse truncation error can provide weaker distortions for modes of a specific parameter range and, therefore, such a method should be considered as more accurate in terms of mode reproduction. In practical simulations this means that the form of a transient given by numerical integration is closer to the true solution than that provided by the method with lower truncation error.

Another consequence of  $\delta_\eta(h)$  and  $\delta_\omega(h)$  being non-monotonous is that decreasing  $h$  does not always improve numerical solution. Using smaller  $h$  usually improves stability and always removes or, at least, weakens numerical oscillations. However, there are ranges of  $h$  where its reducing does not decrease  $\delta_\eta$  and/or  $\delta_\omega$ . Moreover, increasing  $h$  can bring the form of a transient in numerical solution closer to the true form.

## 9 CONCLUSIONS

Analysis of the distortions brings understanding of the accuracy and stability of a method and simulations, unachievable when using truncation error estimations. Character of distortions for the free motion modes can be sophisticated. The more complex (in terms of its scale and parameter range) the simulated network is, the more likely that a distortion analysis would be needed to explain or clarify numerical phenomena observed. Distortion charts based on the approach described here have been built for numerous methods and can be effectively used in network simulations.

Although accuracy analysis of this kind looks like a problem of applied mathematics rather than of electrical engineering, the results obtained seem to be important for practical simulations. The approach discussed in this paper reflects position of an electrical engineer rather than that of a pure mathematician. On the other hand, some of the results obtained are of a highly general character and are applicable not only in electrical networks theory, but also in simulations of mechanical problems, problems of chemical and physical kinetic where numerical integration of ODEs is needed.

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