Alternative Optimal Power Flow Formulations

William Rosehart
Dept of Electrical & Computer Engineering
University of Calgary
Calgary, AB, Canada
rosehart@ieee.org

José A. Aguado
Dept. of Electrical Engineering
University of Málaga
Málaga, Spain
jaguado@uma.es

Abstract - The paper presents an optimal power flow (OPF) formulation using an AC power system model based on current nodal analysis. One of the principle advantages of the proposed formulation is all components of the OPF problem are quadratic, which results in a constant Hessian matrix. The proposed model is compared to the traditional models using interior point point methods applied to test systems.

Keywords - Optimal Power Flow, Power System Modeling, Interior Point Methods

1 Introduction

As open access market principles are applied to power systems, an increased emphasis on using appropriate system modeling will arise. Different transmission line models using either bus voltage magnitudes and phase angles or bus voltages expressed in rectilinear coordinates have been well established in traditional OPF studies, e.g. [1, 2, 3]. In this paper, we propose a novel approach to use current and voltage equations for the OPF problem, versus the traditional active and reactive power equations. Although, nonlinear terms are present in the proposed formulation, the Hessian matrix in the OPF problem is constant. The introduction of additional variables into the power system model using current and voltage based equations has been a strong deterrent to using this type of formulation. The rapid development of computer architectures for large scale problems has lessened this problem. The proposed model is compared to the traditional models using interior point point methods [4] applied to a sample test system.

The paper is structured as follows: In Section 2, the principle symbols used through-out the paper are defined. A brief introduction and review of the optimal power flow problem is presented in Section 3. The proposed system model for the optimal power flow used in the analysis presented in the paper is provided in Section 4. In Section 5, the OPF formulation with the proposed model is outlined. A brief discussion of Interior Point Methods is given in Section 6. An analysis of the results obtained from applying the formulation to the test system is presented in Section 7. Finally, Section 8 summarizes the main contributions of this paper.

2 Nomenclature

Sets:

η	:	Set of all buses (nodes)
η_{nsb}	:	Set of all buses excluding the slack bus
η_{sb}	:	Slack bus
η_g	:	Set of all generator buses
η_l	:	Set of all non-generator buses
η_b	:	Set of all transmission lines
η_t	:	Set of all transformer branches
γ_i	:	Set of all transmission lines connected to bus i
ς_i	:	Set of all transformer branches connected to bus i
ζ_i	:	Set of all loads connected to bus i
ϱ_i	:	Set of all generators connected to bus i

Variables:

$\mathbf{V}$	:	Complex bus voltage vector
$\mathbf{I}$	:	Complex current (branch, generator, load) vector
$\mathbf{a}$	:	Transformer tap ratio vector
$\widetilde{I}_t$	:	*Receiving* bus side complex transformer current
$\widetilde{V}_t$	:	*Receiving* bus side complex transformer voltage

Subscripts:

i, k	:	*indices*
r	:	real component of a variable
m	:	imaginary component of a variable

3 Optimal Power Flow

The OPF problem, introduced in the early 1960's by Carpentier, has grown into a powerful tool for power system operation and planning. In general, the OPF problem is a nonlinear programming (NLP) problem that is used to determine the "optimal" control parameter settings to minimize a desired objective function, subject to certain system constraints [1, 3, 5]. Because of the restructuring of power system utilities [6], different OPF problems are now being considered. The development of numerical analysis techniques and algorithms, particularly Interior Point (IP) methods, allows large and difficult problems to be solved with reasonable computational effort [1, 7]. Power systems are one of the areas where IP methods have been successfully applied (e.g., [1, 8, 9, 10, 11]).

With the introduction of diverse objective functions, the OPF problem represents a variety of optimization problems [3], which includes, for example, active power cost optimization and active power loss minimization [1]. OPF problems are generally formulated as nonlinear programming problems (NLP) as follows:

$$\begin{aligned} \min \quad & \mathbf{f}(\mathbf{x}) \\ \text{s.t. :} \quad & \mathbf{g}(\mathbf{x}) = \mathbf{0} \\ & \underline{\mathbf{h}} \leq \mathbf{h}(\mathbf{x}) \leq \overline{\mathbf{h}} \\ & \underline{\mathbf{x}} \leq \mathbf{x} \leq \overline{\mathbf{x}} \end{aligned} \tag{1}$$

where for most OPF formulations, the components of (1) can be defined as follows:

- The system variables, denoted by the vector $\mathbf{x} \in \Re^q$. Typically, the system variables includes voltage magnitudes and phase angles, generator power levels and transformer tap settings.
- The mapping $\mathbf{f}(\mathbf{x}) : \Re^q \rightarrow \Re$ is a function that is being minimized and can include, for example, total losses in the system and generator costs.
- $\mathbf{g}(\mathbf{x}) : \Re^q \rightarrow \Re^m$ typically represents the load flow equations.
- $\mathbf{h}(\mathbf{x}) : \Re^q \rightarrow \Re^p$ usually stands for transmission line limits, with lower and upper limits represented by $\underline{\mathbf{h}}$ and $\overline{\mathbf{h}}$, respectively. Lower and upper limits of the system variables, $\mathbf{x}$, are given by $\underline{\mathbf{x}}$ and $\overline{\mathbf{x}}$, respectively.

Once formulated, the problem can be solved using Interior Point (IP) methods [2, 12], Sequential Linear Programming (SLP) or Sequential Quadratic Programming (SQP) [11, 13, 14]. SLP and SQP formulations can be solved using well developed Linear and Quadratic Interior Point methods [11, 13]. When applying SLP and SQP methods, convergence has been shown to be dependent on a number of factors, such as good initial conditions and step size control [14].

4 Power System Model

It is proposed in this paper, that the traditional AC power flow equations used to model the system be replaced by a model using currents and voltages expressed in rectangular form. Using the nomenclature defined in Section 2, the complex bus-voltages are defined in rectangular form as:

$$V[i] = V_r[i] + jV_m[i] \qquad \forall i \in \eta \tag{2}$$

where the subscripts r and m are used to denote real and imaginary components, respectively. Similarly, complex branch currents, load currents, and generator currents are defined in rectangular form as:

$$I_b[i] = I_{b_r}[i] + jI_{b_m}[i] \qquad \forall i \in \eta_b \tag{3}$$

$$I_t[i] = I_{t_r}[i] + jI_{t_m}[i] \qquad \forall i \in \eta_t \tag{4}$$

$$I_l[i] = I_{l_r}[i] + jI_{l_m}[i] \qquad \forall i \in \eta_l \tag{5}$$

$$I_g[i] = I_{g_r}[i] + jI_{g_m}[i] \qquad \forall i \in \eta_g \tag{6}$$

where the subscripts b, t, l, and g indicate transmission line branches, transformer branches, loads, and generators respectively.

For each bus in the system, a Kirchhoff's Current Load (KCL) equation is written for the real and imaginary components of the current. This has an analogy with the traditional load flow equations that are normally written for each bus. Therefore, for each bus in the system the following equality constraints must be met:

$$\begin{aligned} 0 = & \sum_{k\in\gamma_i} I_{b_r}[k] + \sum_{k\in\varsigma_i} I_{t_r}[k] \\ & + \sum_{k\in\zeta_i} I_{l_r}[k] - \sum_{k\in\varrho_i} I_{g_r}[k] \quad \forall i \in \eta \end{aligned} \tag{7}$$

$$\begin{aligned} 0 = & \sum_{k\in\gamma_i} I_{b_m}[k] + \sum_{k\in\varsigma_i} I_{t_m}[k] \\ & + \sum_{k\in\zeta_m} I_{l_m}[k] - \sum_{k\in\varrho_i} I_{l_m}[k] \quad \forall i \in \eta \end{aligned} \tag{8}$$

The relationship between the currents defined in equations (7) and (8) and the nodal voltages which will be the principle variables in the OPF formulation are easily defined using traditional techniques. For example, the transmission line branch currents (as illustrated in Figure 1) can be written as:

$$\begin{aligned} \mathbf{I_b} &= (\mathbf{V[c]} - \mathbf{V[d]})\mathbf{Y} \\ &= (\mathbf{V[c]} - \mathbf{V[d]})(\mathbf{G} + \mathbf{jB}) \\ &= ((V_r[c] + jV_m[c]) - (V_r[d] + jV_m[d]))(G + jB) \\ &= ((V_r[c] - V_r[d])G - (V_m[c] - V_m[d])B) \\ &= \quad +j((V_r[c] - V_r[d])B + (V_m[c] - V_m[d])G) \end{aligned} \tag{9}$$

$$I_{b_r} = (V_r[c] - V_r[d])G - (V_m[c] - V_m[d])B \tag{10}$$

$$I_{b_m} = (V_r[c] - V_r[d])B + (V_m[c] - V_m[d])G \tag{11}$$

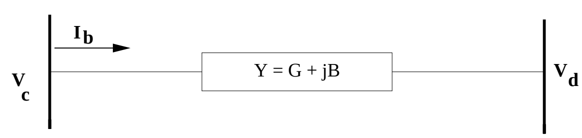

Figure 1: Transmission Line Diagram

For the proposed formulation, the relationship between the complex current associated with transformers and the bus voltages is derived by modeling the transformer as an ideal transformer in series with a series impedance as illustrated in Figure 2. To ensure that power system model remains quadratic, two additional current and voltage variables are introduced as follows:

$$a_i\, I_{t_r} = \widetilde{I_{t_r}}\,, \qquad a_i\, I_{t_m} = \widetilde{I_{t_m}} \qquad \forall i \in \eta_t \tag{12}$$

$$V_{c_r} = a_i\, \widetilde{V_{c_r}}\,, \qquad V_{c_m} = a_i\, \widetilde{V_{c_m}} \qquad \forall i \in \eta_t \tag{13}$$

where a_i is the off nominal tap setting of transformer $i \in \eta_t$ and $\widetilde{V_{c_{r/m}}}$ and $\widetilde{I_{t_{r/m}}}$ are introduced to simply the equations describing the current voltage relationship at the *sending* and *receiving* buses.

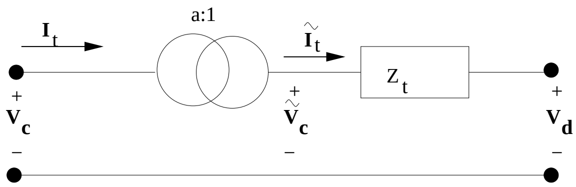

Figure 2: Transformer Single Line Diagram

A *pi* model is used to incorporate the shunt capacitance to ground. Using equations (12) and (13), Figure 2, and the transmission line equations (10) and (11), the voltage/current relationship for the transformer branches can be written as follows:

$$\widetilde{I_r[c]} = (\widetilde{V_r[c]} - V_r[d])G - (\widetilde{V_m[c]} - V_m[d])B \quad (14)$$

$$\widetilde{I_i[c]} = (\widetilde{V_r[c]} - V_r[d])B + (\widetilde{V_m[c]} - V_m[d])G \quad (15)$$

$$I_r[d] = (V_r[d] - aV_r[c])G - (V_m[d] - aV_m[c])B \quad (16)$$

$$I_r[d] = (V_r[d] - aV_r[c])B + (V_m[d] - aV_m[c])G \quad (17)$$

Finally, the relationship between the real and reactive power, and the complex current for generators and loads is determined using the fundamental definition of complex power as follows:

$$\begin{aligned} \mathbf{S} &= P + jQ \quad (18)\\ (V_r + jV_m)(I_r + jI_m)^* &= P + jQ \\ (V_r + jV_m)(I_r - jI_m) &= P + jQ \end{aligned}$$

Therefore, for generators and PQ loads:

$$P = V_r I_r + V_m I_m \quad (19)$$

$$Q = -V_r I_m + V_m I_r \quad (20)$$

The above equality equations, particularly equations (7, 8, 10-13, 16, 17, 19, 20) are used to *replace* the traditional power flow equations. It should be noted that the above equations are strictly linear or quadratic. There are no *higher order* terms or sinusoidal expressions.

Traditional limits, such as voltage limits can be incorporated by defining the voltage magnitude at a particular bus in terms of the complex voltage as follows:

$$V_{min}[i]^2 \le V_r[i]^2 + V_m[i]^2 \le V_{max}[i]^2 \quad \forall i \in \eta \quad (21)$$

Furthermore, transmission line limits are defined in terms of the actual current limit, versus the more traditional approximation of defining transmission line limits in terms of power.

All the functions in the above formulation are quadratic. This presents the following numerical advantages when incorporating this equations to model the power system

1. The Taylor series expansion of a quadratic function $\mathbf{f}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x}$ terminates at the second-order term without truncation error, i.e.,

$$\mathbf{f}(\mathbf{x}^k + \Delta\mathbf{x}) = \mathbf{f}(\mathbf{x}^k) + (\mathbf{x}^k)^T\mathbf{A}\Delta\mathbf{x} + \frac{1}{2}\Delta\mathbf{x}^T\mathbf{A}\Delta\mathbf{x} \quad (22)$$

2. The Hessian of $\mathbf{f}(\mathbf{x})$ is constant (where the Hessian is defined as $\mathbf{H}(x) = \mathbf{A}$).

3. The second-order terms in (22) can be easily evaluated.

These advantages are the same as those presented in [2] for a quadratic model. In [2], a quasi-quadratic power system model is presented using the traditional power flow equations with rectangular voltages.

5 OPF Formulation

In this section the power system model presented in Section 4 is used to formulate an OPF problem of the form given by equation (1). For simplicity, the objective will be set to minimize the active power costs of the generators in the system. This formulation, can be easily changed to incorporate bids from various generators, as was in done in [15]. The objective function is written as:

$$\mathbf{f}(\mathbf{x}) = \sum_{i \in \eta_g} c_0 + c_1 P_i + c_2 P_i^2 \quad (23)$$

where the constants c_0, c_1, c_2 represent the coefficients associated with the quadratic operating cost model of the generators [1]. In the proposed model, the vector of system variables, $\mathbf{x}$, is composed of the following:

$$\mathbf{x} = \begin{bmatrix} v_r[i] & i \in \eta \\ v_m[i] & i \in \eta_{nsb} \\ I_{b_r}[i] & i \in \eta_b \\ I_{b_m}[i] & i \in \eta_b \\ I_{l_r}[i] & i \in \eta_l \\ I_{l_m}[i] & i \in \eta_l \\ I_{g_r}[i] & i \in \eta_g \\ I_{g_m}[i] & i \in \eta_g \\ P_g[i] & i \in \eta_g \\ Q_g[i] & i \in \eta_g \\ a[i] & i \in \eta_t \\ \widetilde{I_{t_r}}[i] & i \in \eta_t \\ \widetilde{I_{t_m}}[i] & i \in \eta_t \\ \widetilde{V_{t_r}}[i] & i \in \eta_t \\ \widetilde{V_{t_m}}[i] & i \in \eta_t \end{bmatrix} \quad (24)$$

where v_m at the slack bus is constant and equal to zero.

The set of equality constraints based on the rectangular current and voltage equations presented in Section 4 are as follows:

$\mathbf{g(x)} =$

$$\begin{bmatrix} \sum_{k\in\gamma_i} I_{b_r}[k] + \sum_{k\in\zeta_i} I_{l_r}[k] - \sum_{k\in\varrho_i} I_{l_r}[k] & i \in \eta \\ \sum_{k\in\gamma_i} I_{b_m}[k] + \sum_{k\in\zeta_m} I_{l_m}[k] - \sum_{k\in\varrho_i} I_{l_m}[k] & i \in \eta \\ (V_{R_l} - V_{R_k})G - (V_{I_l} - V_{I_k})B - I_R & i \in \eta_b \\ (V_{R_l} - V_{R_k})B + (V_{I_l} - V_{I_k})G - I_R & i \in \eta_b \\ a\,\overline{V_l} - V_l & i \in \eta_t \\ a\,I_l - \overline{I_l} & i \in \eta_t \\ (\overline{V_{R_l}} - V_{R_k})G - (\overline{V_{R_l}} - V_{I_k})B - \overline{I_{i_R}} & i \in \eta_t \\ (\overline{V_{R_l}} - V_{R_k})B + (\overline{V_{R_l}} - V_{I_k})G - \overline{I_{i_I}} & i \in \eta_t \\ V_R I_R + V_I I_I - P[i] & i \in \eta_g \\ -V_R I_I + V_I I_R - Q[i] & i \in \eta_g \end{bmatrix} \quad (25)$$

Finally, the inequality constraints used to represent the bus voltages, transmission line and transformer thermal limits, generator power limits, and transformer tap limits are as follows:

$\mathbf{h(x)} =$

$$\begin{bmatrix} V_{min}[i]^2 \le V_r[i]^2 + V_m[i]^2 \le V_{max}[i]^2 & \forall i \in \eta \\ P_{min}[i] \le P[i] \le P_{max}[i] & \forall i \in \eta_g \\ Q_{min}[i] \le Q[i] \le Q_{max}[i] & \forall i \in \eta_g \\ I_{min}[i]^2 \le I_r[i]^2 + I_m[i]^2 \le I_{max}[i]^2 & \forall i \in \eta_g \\ I_{min}[i]^2 \le I_r[i]^2 + I_m[i]^2 \le I_{max}[i]^2 & \forall i \in \eta_b \\ I_{min}[i]^2 \le I_r[i]^2 + I_m[i]^2 \le I_{max}[i]^2 & \forall i \in \eta_t \\ a_{min}[i] \le a[i] \le a_{max}[i] & \forall i \in \eta_t \end{bmatrix} \quad (26)$$

Additional constraints, can be easily incorporated into the above model if required. The nonlinear OPF problem formulated above is non-convex.

As discussed in Section 4, the above OPF formulation is quadratic, which can enhance the numerical performance when solving the optimization problem [2]. The disadvantage of the proposed model, is the increased number of variables introduced into the problem. It is proposed that enhancements in interior point methods, limits the disadvantage of introducing additional variables, since the objective and constraints are quadratic.

6 Interior Point Methods

Optimization techniques, in particular Interior Point Methods, have been repeatedly enhanced over the last decade [4, 16]. The development of numerical analysis techniques and algorithms, particularly Interior Point (IP) methods, allows large and difficult problems to be solved with reasonable computational effort [1, 7]. Power systems are one of the areas where IP methods have been successfully applied (e.g., [1, 8, 9, 10, 11]). One of the more commonly applied IP methods is the Primal-Dual IP method [2, 8].

For the purpose of the research presented in this paper, the Nonlinear Interior Point program LOQO [17] is used to solve the proposed OPF formulations. LOQO is based on a sequential quadratic primal-dual interior point method [17]. The modeling language AMPL [18] was used to formulate the problem.

7 Numerical Simulations

The rectangular current and voltage based OPF formulation is tested on a system based on the IEEE 30 bus test system [19]. The first stage in implementing the proposed OPF formulation was to develop a load flow solution to demonstrate feasibility. This was done using Matlab [20]. The results when solving the rectangular current and voltage *load flow* indicated the model was robust and numerically stable.

For the test system, the OPF was solved using both a traditional AC power system model based on voltage magnitudes and phase angles as well as the proposed formulation. The solution from each formulation was tested for feasibility using the other formulation. To improve the numerical performance of the proposed formulation, an additional constraint that the real component of the bus voltages must be positive was introduced.

Figure 3 is a plot of the bus voltage magnitudes for the solution obtained using the traditional AC model and the proposed model. From this figure, it can be seen, that the two methods to model the system are equivalent and converge to the same optimal solution, as expected.

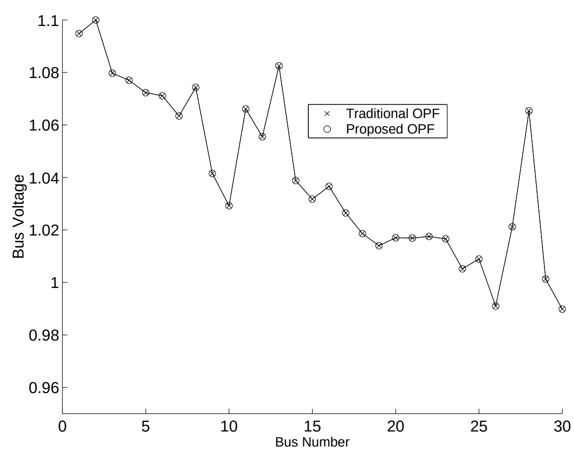

Figure 3: Bus Voltage Magnitudes for the Traditional and Proposed OPF Formulations

The running time of the traditional method was approximately fifty percent faster than the proposed method. This was attributed to the fact that the proposed formulation required 36 iterations to converge versus 20 iterations for the traditional formulation. The time to compute the Hessian matrix for the proposed formulation was three times faster than the traditional method.

8 Conclusions

In this paper, an OPF formulation was proposed based on a rectangular current nodal analysis versus the traditional power flow. The advantage of the proposed formulation is the Hessian matrix of the optimization problem is convex. This resulted in significantly faster times to calculate the Hessian matrix. The greatest disadvantage of the proposed method, is the increased number of variables versus one of the traditional models. Further numerical studies will be performed on larger system to better determine the general characteristics of the proposed OPF formulation.

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