ANALYSIS OF NOISY VOLTAGE SIGNAL WITH A HIGH RESOLUTION OF FREQUENCY FOR THE CLOSING OF TRANSMISSION LINES

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Abstract – This paper presents an auto – reclosing technique for compensated transmission lines, which uses the Prony Method for the determination of the optimal switching moment. The influence of the signal noise to this algorithm is reduced with the help of the singular value decomposition. The mathematical basics for both techniques are explained and the results of computer simulations are shown.

***Keywords:** optimal switching moment, Prony Method, singular value decomposition*

1 INTRODUCTION

A reduction of the switching overvoltage and inrush current is possible by closing transmission lines in the optimal switching moment [1][2]. For this it is necessary to analyze the voltage over the main switching contacts, to predict the voltage shape and to identify the optimal switching moment. The parameters of the voltage signal differ in frequency, damping ratio and amplitude. The voltage of the bus-bar has the system frequency and constant amplitude. The prediction of the parameter of this voltage shape is simple. That's the reason, only why the voltage of the line side is observed. The waveform of this voltage depends on the design of the transmission line and the connected equipment (e.g. measuring transformer). The voltage signals during auto – reclosings of transmission lines, which are compensated by shunt reactors are discussed in this paper. The resulting line-to-earth voltage is in this case defined by the resonant circuit (shunt reactor and system capacitance) and the induced voltage from the other phases. The voltage of the neighboring phases differs in frequency, because the rate of compensation is practically never the same in all three phases. In addition a noise is superimposed. The voltage transformer and the digitalization are the reasons for this noise. The real time processing ability is a further criterion for the algorithm. In this example the maximum calculation time is limited by the interruption time of the auto-reclosing cycle, which depends on the voltage level.

Up to now the method of pattern recognition is used for this calculation [3]. This method only analyzes the envelope curve of the voltage signal, if the signal consists of more than one frequency [1], [2]. Therefore it is not possible to reclose in the zero crossing, which is the optimal moment for this switching task. To fulfill this demand it is necessary to calculate the individual signal parameters, which is impossible by pattern recognition.

The Fourier transformation as the classical arithmetical method for the estimation of the parameters of signal components has to be excluded. The reasons are the low resolution in the frequency domain and the necessary long evaluation window for low frequency signal components. Furthermore is it not possible to calculate existing damping factors with the Fourier transformation.

The Prony Method represents a good solution to analyze the voltage signals. The method composes the voltage signal by a sum of sinusoidal and exponential damped functions. The parameters of these functions are the amplitude, the damping factor, the frequency and the phase shift. The lowest recognizable frequency is independent of the evaluation window of the Prony Method. Also signal components with periods greater than the evaluation windows are correctly calculated. Another advantage of the Prony Method is a high resolution in the frequency domain. Signals with small frequency differences will be exactly determined. Unfortunately the algorithm responds incorrect results already in present of a low noise level. Especially the damping factor will be wrong estimated and therefore prediction of voltage will be incorrect. A solution for this problem is the Prony Method in combination with the method of singular value decomposition (SVD).

The theory of the least square Prony Method will be explained and the procedure will be demonstrated by the prediction of voltage signals. In the following chapter the basics of the singular value decomposition method are shown and improved results are explained. In the last chapter the experiences and results of simulations are documented.

2 LEAST SQUARE PRONY METHOD

The N data samples $x[1],\dots,x[n],\dots,x[N]$ with the constant sample interval T are the starting point. This vector can be also described with the following summation with a p-term complex exponential model.

$$x[n] = \sum_{k=1}^{p} A_k \exp[(\alpha_k + j2\pi f_k)(n-1)T + j\theta_k] \quad (1)$$

In this equation A_k is the amplitude, α_k the damping factor in seconds^{-1}, f_k the sinusoidal frequency in Hz and θ_k the sinusoidal initial phase in radians. If the data samples are real, then the complex exponential must occur in complex conjugate pairs of equal amplitude. The eq. (1) may expressed in the form

$$x[n] = \sum_{k=1}^{p} \underline{h}_k \underline{z}_k^{n-1}\,, \tag{2}$$

where the complex constants $\underline{h}_k$ and $\underline{z}_k$ are defined as

$$\underline{h}_k = A_k \exp(j\theta)\,, \tag{3}$$

$$\underline{z}_k = \exp[(\alpha_k + j2\pi f_k)T]\,. \tag{4}$$

The goal for the algorithm is to minimize the squared error over all sampled data with respect to the complex parameters and the number of exponents.

No analytic solution exists for the minimization [4]. The real time requirements and the high demand on the processor power are the reason, that iterative algorithm can't be used. The least – square Prony Method represent a sub optimum minimization for that problem.

The number of data points N is in practice usually greater than the minimum number needed to fit a model of p exponential ($N > 2p$). The data sequence can be approximated as an exponential sequence (see eq. (2)). This equation can also represent in matrix form for $1 \le n \le N$.

$$\begin{pmatrix} \underline{z}_1^0 & \underline{z}_2^0 & \cdots & \underline{z}_p^0 \\ \underline{z}_1^1 & \underline{z}_2^1 & \cdots & \underline{z}_p^1 \\ \vdots & \vdots & \ddots & \vdots \\ \underline{z}_1^{N-1} & \underline{z}_2^{N-1} & \cdots & \underline{z}_p^{N-1} \end{pmatrix} \cdot \begin{pmatrix} \underline{h}_1 \\ \underline{h}_2 \\ \vdots \\ \underline{h}_p \end{pmatrix} = \begin{pmatrix} x[1] \\ x[2] \\ \vdots \\ x[N] \end{pmatrix} \tag{5}$$

$$\underline{\mathbf{Z}} \cdot \underline{\mathbf{h}} = \mathbf{x}$$

If $\underline{\mathbf{Z}}$ are known, then eq. (5) represent a set of linear equations to solve the unknown amplitude vector $\underline{h}$. Prony's contribution was the discovery of a method, which determinates the unknown matrix $\underline{\mathbf{Z}}$. According Prony is eq. (2) the solution of a homogeneous – linear difference equation. The roots of this unknown difference equation are the $\underline{z}_k$ exponents. First a polynomial $\underline{\phi}$ will be defined.

$$\underline{\phi}[\underline{z}] = \prod_{k=1}^{p} (\underline{z} - \underline{z}_k) \tag{6}$$

This polynomial can also be represented by a summation [5].

$$\underline{\phi}[\underline{z}] = \sum_{m=0}^{p} \underline{a}[m] \underline{z}^{p-m} \tag{7}$$

The first value for the complex coefficient $\underline{\mathbf{a}}$ will define as $\underline{a}[0]=1$. Shifting the index from n to n-m in eq. (2) and multiplying with the parameter $\underline{a}[m]$ are the next steps.

$$\underline{a}[m]x[n-m] = \underline{a}[m] \sum_{k=1}^{p} \underline{h}_k \underline{z}_k^{n-m-1} \tag{8}$$

The next operations are some mathematical transformations:

1. Forming similar products $(\underline{a}[0]x[n], \ldots, \underline{a}[m-1]x[n-m+1])$
2. Summation of this products
3. Substitution $\underline{z}_k^{n-m-1} = \underline{z}_k^{n-p-1} \underline{z}_k^{p-m}$

A detailed discussion of these steps are in [4].

$$\sum_{m=0}^{p} \underline{a}[m]x[n-m] = \sum_{k=1}^{p} \left[\underline{h}_k \underline{z}_k^{n-p-1} \sum_{m=0}^{p} \underline{a}[m] \underline{z}_k^{p-m} \right] = e[n] \tag{9}$$

The eq. (9) is valid for $p+1 \le n \le N$ and the searched linear difference equation whose homogeneous solution is given by eq. (2). The right summation (first row) is the polynomial defined by eq. (7). The term $e[n]$ represents the exponential approximation error. The left summation in eq. (9) is also called as the forward linear prediction error equation with the linear prediction parameter $\underline{a}[m]$ [4]. This parameter may be selected as those that minimize the linear prediction squared error $\rho = \sum_{n=p+1}^{N} |e[n]|^2$. So the eq. (9) can also be represented in matrix form.

$$\begin{pmatrix} \underline{x}[p+1] & \cdots & \underline{x}[1] \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \underline{x}[N] & \cdots & \underline{x}[N-p] \end{pmatrix} \cdot \begin{pmatrix} \underline{a}[0] \\ \vdots \\ \vdots \\ \underline{a}[p] \end{pmatrix} = \begin{pmatrix} e[p+1] \\ \vdots \\ \vdots \\ e[N] \end{pmatrix} \tag{10}$$

$$\mathbf{T} \cdot \underline{\mathbf{a}} = \mathbf{e}$$

The normal equation to minimize squared error is:

$$\mathbf{T}^H \mathbf{T} \underline{\mathbf{a}} = \begin{pmatrix} \rho \\ 0 \end{pmatrix}. \tag{11}$$

The operand $\mathbf{T}^H$ mean the complex conjugating all the elements of matrix $\mathbf{T}$ and the following transposing of the elements. The eq. (11) is over - determined because the first row are known ($\underline{a}[0]=1$). So this first row can be eliminated and the first column of the matrix product is the solution vector. Now it is possible to calculate $\underline{z}_1, \ldots, \underline{z}_p$ by the determination of $\underline{\mathbf{a}}$ and following polynomial factoring. The calculation of the frequency and the damping ratio is carried out by eq. (4). The eq. (5) had to be converted in eq. (12) to minimize the squared error of the amplitude vector $\underline{h}_k$. A deduction for that minimizing problem is in [4].

$$(\underline{\mathbf{Z}}^H \underline{\mathbf{Z}})\ \underline{\mathbf{h}} = (\underline{\mathbf{Z}}^H \mathbf{x}) \tag{12}$$

The calculations of amplitude and phase shift are now possible with the help of eq. (3).

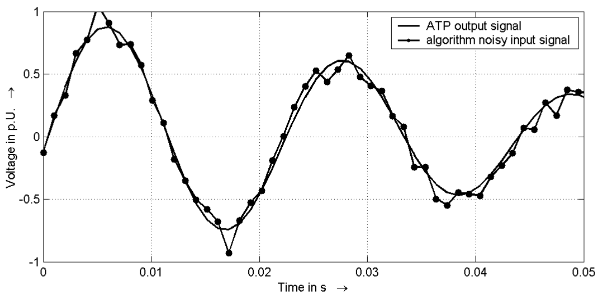

Figure 1 ATP – output signal and noisy input signal of the Prony algorithm

As an example the line voltage of phase a after a simulated auto-reclosing of a 500 kV compensated overhead transmission line will be analyzed. The transient simulation was carried out with the program ATP. The settings of the compensation were in phase a and b 80% and in phase c 75% (see introduction – practical compensation). The amplitude of the system voltage is the reference value for the input vector. The voltage of the line side are superpose with additive noise. The variance of noise ratio is 0.005 (see Figure 1). In this example the input vector of the algorithm has a size of N=250 samples and a sample interval of T=0.001s.

The Figure 2 represent the results of the calculation with an exponential model of the size p=4. In the upper diagram the input signal itself and his estimation are represented. In the lower diagram the same signals are shown during a longer time interval. The estimated voltage is calculated with the parameter of the first data window.

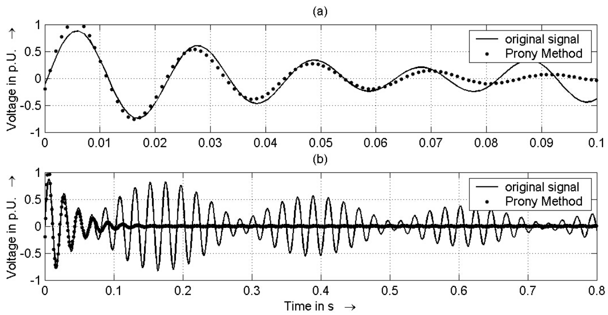

Figure 2 (a) sampled data and Prony Method for the period of sampling; (b) sampled data and Prony Method for the period of sampling and period of estimation in to the future

The results are insufficient. Only one part of the original signal is correctly approximated, but still with a false damping ratio. The results in other simulations, e.g. with an other compensation, are similar. The reason for the bad results is that there is no noise model in the Prony Method implemented [4][6].

In the next section a suggestion of reducing the influence of the noise in the data matrix is discussed to improve the results of the Prony Method.

3 PRONY METHOD AND SINGULAR VALUE DECOMPOSITION (SVD)

If the input vector contains no noise (exponential approximation error are zero), the forward linear prediction eq. (9) can be written as

$$\sum_{m=0}^{p} \underline{a}[m]x[n-m]=0 \,. \tag{13}$$

$$\underline{A}(\underline{z})=\sum_{m=0}^{p} \underline{a}[m]z^{p-m} \tag{14}$$

The pertinent characteristic polynomial $\underline{A}(\underline{z})$ (see eq. (7)) has roots at $\underline{z}_k$=exp($\underline{s}_k$)=exp([α_k+$j2\pi f_k$]T). The same exponentials may be generated in reverse time by the backward linear prediction.

$$\sum_{m=0}^{p} \underline{b}[m]x[n-p+m]=0 \tag{15}$$

The characteristic polynomial will be defined as

$$\underline{B}(z)=\sum_{m=0}^{p} \underline{b}^*[m]\underline{z}^{p-m} \,. \tag{16}$$

The roots $\underline{z}_k=\exp\left(-\underline{s}_k^*\right)=\exp\left(\left[-\alpha_k+j2\pi f_k\right]\,T\right)$ are formed from the conjugated complex backward linear prediction coefficients. The roots of the forward linear characteristic polynomial fall inside the unit $\underline{z}$-plane circle for a decaying factor of damping ratio. The roots of the backward linear characteristic polynomial fall outside, because this is a fast growing exponential function. This effect is all the more stronger, the more the absolute damping ratio is higher. The roots for the forward and backward polynomial for the example from Figure 2 (analyze without noise) are represent in Figure 3.

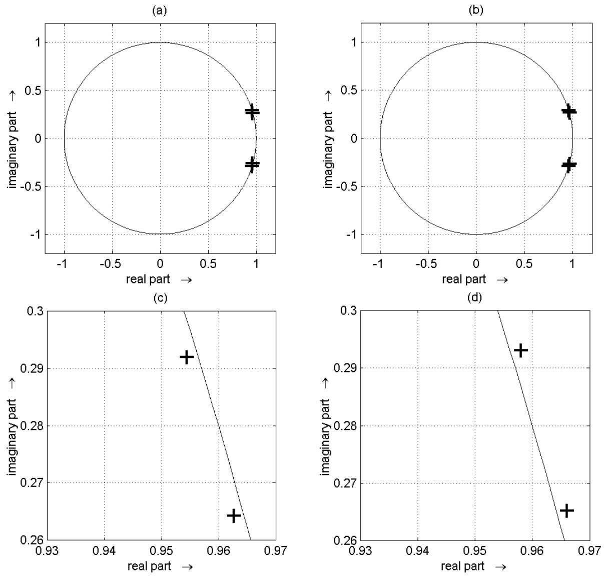

Figure 3 (a) $\underline{A}(\underline{z})$, p=4, no noise, (b) $\underline{B}(\underline{z})$, p=4, no noise, (c) $\underline{A}(\underline{z})$, p=4, no noise, zoom, (d) $\underline{B}(\underline{z})$, p=4, no noise, zoom

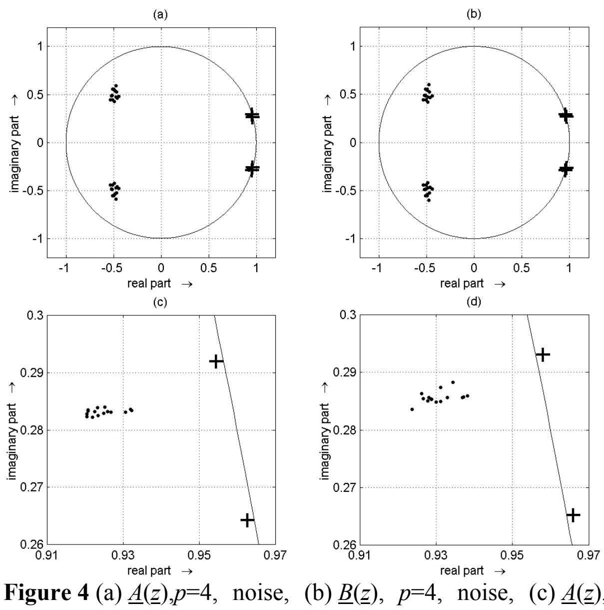

Figure 4 (a) $\underline{A}(\underline{z})$,$p$=4, noise, (b) $\underline{B}(\underline{z})$, p=4, noise, (c) $\underline{A}(\underline{z})$, p=4, noise, zoom, (d) $\underline{B}(\underline{z})$, p=4, noise, zoom

The roots of the example (analyze now with noise) from chapter 2 are shown in Figure 4. This example with the additive noise was fifteen times repeated (points). Also the true zeros are pictured (cross). It is visible that only one calculated root is in the proximity of the true roots (diagram (c) and (d)). This applies however only to the forward polynomial. For the backward polynomial these roots do not fall outside of the unit circle. If the true damping ratio increases or the noise decreases, this effect will not occur [7]. The false calculated roots correspond to a signal with a high frequency and strong damping ratio. The Prony Method tries to emulate the noise in the input signal with this signal component. When the statistic parameters of a stationary random process do not change even though the time is reversed, the roots of this backward polynomial fall also inside the unit circle.

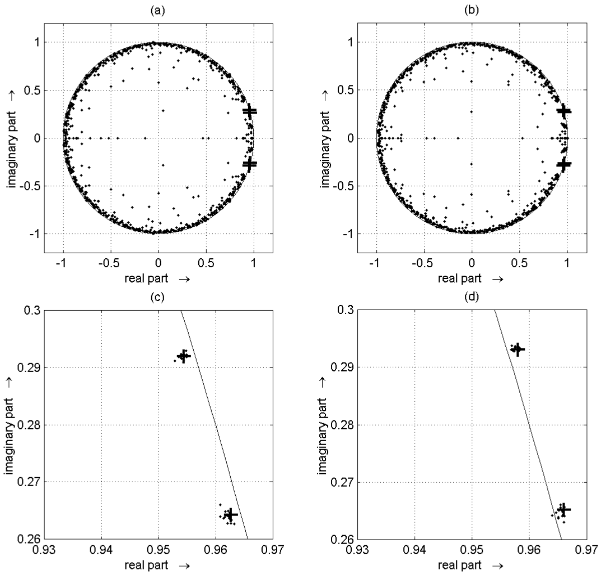

Figure 5 (a) $\underline{A}(\underline{z})$, p=64, noise, (b) $\underline{B}(\underline{z})$, p=64, noise, (c) $\underline{A}(\underline{z})$, p=64, noise, zoom, (d) $\underline{B}(\underline{z})$, p=64, noise, zoom

In the literature it is recommended [7] to increase the linear prediction order p to get better results. The variable p must keep the condition $p+1 \le n \le N$ (see eq. (9)). The results of the estimation with p=64 for fifteen investigations are shown in Figure 5. The true position of the roots is estimated correctly with this high order of p (diagram c and d). But it is not possible to distinguish between the true roots and the false roots, if the correct position of the roots is unknown. The roots of the noise are located on a circular path within the unit circle for forward and backward polynomial. This is the representation of an exponential model, which approximates the noise. Pitiable the radius of the circular path oscillates between the individual investigations. The distinction between true and noise roots should be possible with a higher damping ratio. In this case the position of the roots from the backward polynomial are clearly outside of the unit circle.But this criterion is not able for this kind of signal.

The application of singular value decomposition (SVD) can provide a further improvement. The forward and backward prediction error can be rewritten analog to eq. (10) as

$$\mathbf{T}_p^f \underline{\mathbf{a}}_p^f = -\mathbf{x}_p^f + \mathbf{e}_p^f$$
$$\mathbf{T}_p^b \underline{\mathbf{a}}_p^b = -\mathbf{x}_p^b + \mathbf{e}_p^b. \qquad (17)$$

Thereby $\mathbf{T}_p^f$ and $\mathbf{T}_p^b$ are the data matrix, the forward and backward linear prediction coefficient vector are $\underline{\mathbf{a}}_p^f$ and $\underline{\mathbf{a}}_p^b$, the forward and backward linear prediction error vector are $\mathbf{e}_p^f$ and $\mathbf{e}_p^b$ and the data vectors are $\mathbf{x}_p^f$ and $\mathbf{x}_p^b$. This matrices and vectors are defined as

$$\mathbf{T}_p^f = \begin{pmatrix} x[p] & \dots & x[1] \\ \vdots & & \vdots \\ x[N-1] & \dots & x[N-p] \end{pmatrix},$$
$$\mathbf{T}_p^b = \begin{pmatrix} x[p+1] & \dots & x[2] \\ \vdots & & \vdots \\ x[N] & \dots & x[N-p+1] \end{pmatrix},$$
$$\mathbf{x}_p^f = \begin{pmatrix} x[p+1] \\ \vdots \\ x[N] \end{pmatrix}, \quad \mathbf{x}_p^b = \begin{pmatrix} x[1] \\ \vdots \\ x[N-p] \end{pmatrix}, \qquad (18)$$
$$\underline{\mathbf{a}}_p^f = \begin{pmatrix} a^f[1] \\ \vdots \\ a^f[p] \end{pmatrix}, \quad \underline{\mathbf{a}}_p^b = \begin{pmatrix} a^b[p] \\ \vdots \\ a^b[1] \end{pmatrix}.$$

The rectangular data matrices can be fragmented in positive real numbers σ, which are called singular values, and two eigenvectors $\mathbf{u}_n$ and $\mathbf{v}_n$. After [8] the data matrix have the following singular value decomposition.

$$\mathbf{T}_p^f = \sum_{n=1}^{p} \sigma_n^f \mathbf{u}_n^f \left(\mathbf{v}_n^f\right)^H$$
$$\mathbf{T}_p^b = \sum_{n=1}^{p} \sigma_n^b \mathbf{u}_n^b \left(\mathbf{v}_n^b\right)^H \qquad (19)$$

The singular values can arranged into the order $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_p > 0$. According to [4], the m largest singular values and the pertinent m eigenvectors are associated to the m searched exponential components. The other p-m singular values and eigenvectors span the noise zone. So the rank of the data matrices and the ratio of noise in the data matrices can be reduced. This reduction of noise is shown in Figure 6.

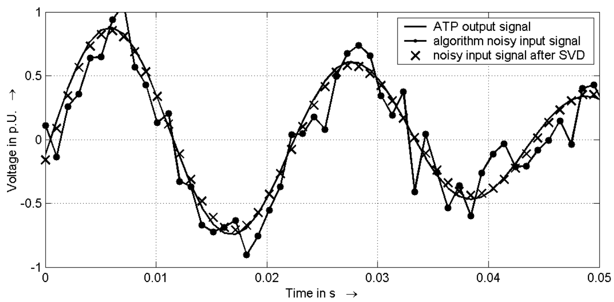

Figure 6 ATP – output signal, noisy input signal of the algorithm and the noisy input vector after SVD

The new data matrices will be defined as

$$\hat{\mathbf{T}}_p^f = \sum_{n=1}^{m} \sigma_n^f \mathbf{u}_n^f \left(\mathbf{v}_n^f\right)^H$$
$$\hat{\mathbf{T}}_p^b = \sum_{n=1}^{m} \sigma_n^b \mathbf{u}_n^b \left(\mathbf{v}_n^b\right)^H. \quad (20)$$

The prediction coefficients will be calculated with the new data matrices.

$$\underline{\mathbf{a}}_p^f = -\left(\hat{\mathbf{T}}_p^f\right)^{\#} \mathbf{x}_p^f$$
$$\underline{\mathbf{a}}_p^b = -\left(\hat{\mathbf{T}}_p^b\right)^{\#} \mathbf{x}_p^b \quad (21)$$

The operand # means the pseudoinverse of the new data matrices, which is defined as

$$\left(\hat{\mathbf{T}}_p^f\right)^{\#} = \sum_{n=1}^{m} \left(\sigma_n^f\right)^{-1} \mathbf{v}_n^f \left(\mathbf{u}_n^f\right)^H$$
$$\left(\hat{\mathbf{T}}_p^b\right)^{\#} = \sum_{n=1}^{m} \left(\sigma_n^b\right)^{-1} \mathbf{v}_n^b \left(\mathbf{u}_n^b\right)^H. \quad (22)$$

The order p must be lying in the range $m \le p \le N-m$, otherwise the rank of $\mathbf{T}_p^f$ and $\mathbf{T}_p^b$ are smaller then m. Now the determination of the coefficient $\underline{z}_1,\ldots,\underline{z}_p$ with polynomial factoring is possible.

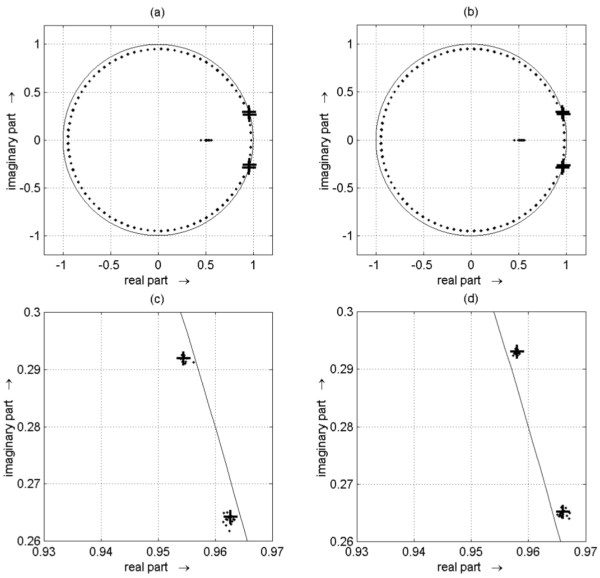

Figure 7 (a) $\underline{A}(\underline{z})$, p=64, m=4, noise, (b) $\underline{B}(\underline{z})$, p=64, m=4, noise, (c) $\underline{A}(\underline{z})$, p=64, m=4, noise, zoom, (d) $\underline{B}(\underline{z})$, p=64, m=4, noise, zoom

The new roots of the noise (see Figure 7) are always within the circle for the forward and backward polynomial, but the radius of the circular path does not oscillate longer between the individual investigations. The position of the true roots is correctly calculated. The true and the calculated roots are nearly correctly.

Now it is possible to define a simple algorithm, which can distinguish between true and noise roots. This and the results of more computer simulations are the topic of the next chapter.

4 RESULTS OF COMPUTER SIMULATION

The number of searched exponential components is normally unknown (variable m from eq. (20)). The sorted value for the singular values (the same example as in the previous sections) for fifteen repetitions are shown in Figure 8. The singular values for the searched signal components are much greater than the value for the noise. For this reason a limit of noise was defined. All values over the limit are accepted and the other are rejected.

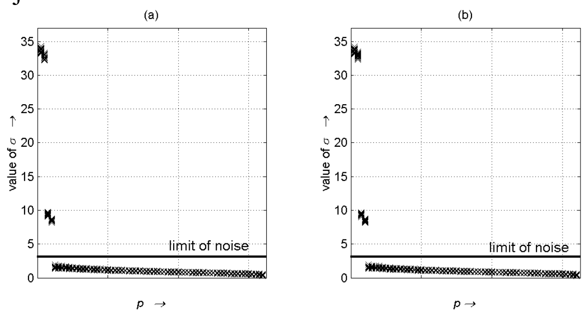

Figure 8 (a) value of σ for the forward prediction (b) value of σ for the backward prediction

The storage of the variable m is useful. Naturally a good prediction of the voltage is now possible. But for the distinction between roots of the true signal and the roots of noise two SVD are necessary. This consumes a lot of computing time. The roots of $\underline{z}_k$ (forward polynomial) and the calculated amplitudes (see eq. (3)) are shown in Figure 9. The amplitudes of the wanted signal components are much greater than the amplitudes of the noise signal components.

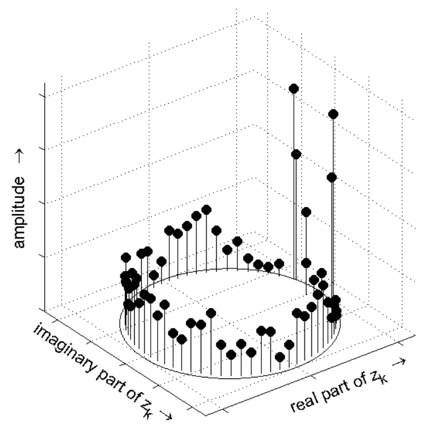

Figure 9 roots of $\underline{z}_k$ and the pertinent amplitudes of the forward polynomial

The calculation of the future voltage should take place only with this $\underline{h}_k$ and $\underline{z}_k$ parameters which corresponding with the m largest amplitudes. So it is only one SVD necessary and also the calculation matrices for the future voltage will be smaller. The computing time can be reduced. The results with this algorithm for the example from Figure 2 are shown in Figure 10. The quality of the estimation for the future signal is now sufficient. A selection of a zero crossing as optimal switching moment is possible.

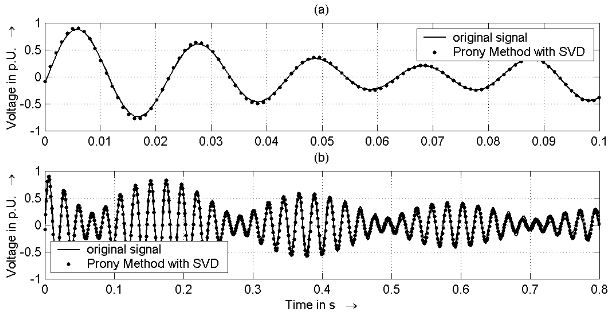

Figure 10 (a) sampled data and Prony Method with SVD for the period of sampling; (b) sampled data and Prony Method with SVD for the period of sampling and period of estimation in to the future

The validity of this algorithm was verified with simulation. The parameter compensation level in all three phases, level of noise and type of fault was modified for this study. The transient process was simulated with ATP and the following calculation has been carried out with MATLAB. The estimated future voltage was compared with the original voltage from ATP for the rating. The best results were obtained with a sample rate, which is situated in the range of 1kHz. The length of sampling is dependent on the level of noise, with a rising noise level the length of the calculation window had to be enlarged. The same statement is correct for the relation between number of signals in the input vector (number of signals are dependent on accuracy of compensation in all three phases) and the length of calculation window. The value of p (see eq. (19)) should be selected between 50 and 80. A smaller value may be accepted if the noise level and the number of signal components are smaller.

5 CONCLUSION AND PERSPECTIVE

In this paper the application of the Prony Method in combination with the singular value decomposition for an optimal auto – reclosing technique of compensated transmission lines in real noisy enviroment had been shown. The use of this mathematical algorithm reduce the influence of noise in the input vector. It could be shown that the prediction of the future voltage with the Prony Method is possible.

The next goal is the reduction of the calculation time and so the verification for real – time processing of the algorithm. For this purpose the algorithm should implement to a evaluation board with a DSP. The transient signals will create again in ATP and will be transferred to the board in real time. The reaction of the board will be recorded and analyzed.

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