Abstract— This paper presents fundamental concepts associated with the method of Shifted Frequency Analysis (SFA) for modelling electric circuit transients at frequencies close to the fundamental rated power frequency (50Hz or 60Hz). In a Smart Grids scenario, where the inertia of the generating sources is much less than at the transmission system level, electric transients simulation tools like the Electromagnetic Transients Program - EMTP become particularly important to assess the impact of distributed generation and voltage and frequency dynamics. With SFA implemented in the EMTP, very accurate results can be achieved with relatively large time steps that can trace the envelopes of transient voltages and currents around the fundamental frequency. If required, instantaneous values can be obtained directly from the “dynamic phasor solution” obtained with SFA. This paper presents a rigorous mathematical derivation of SFA based on the Hilbert transform. Error analysis is performed in terms of the integration time step size and the width of the frequency band around the fundamental frequency. A test case illustrates the main modelling concepts.

Keywords— EMTP, Shifted Frequency Analysis, Computer Modelling, Dynamic Phasors, Time Step Size, Power System Dynamics, Smart Grids.

I. INTRODUCTION

This paper reviews the essential modelling concepts used by the Shifted Frequency Analysis (SFA) technique [1]-[5]. SFA is based on finding an EMTP solution [6] in the Shifted Frequency Domain (SFA domain) and then bringing this solution back into the real time domain. Shifting by 60 Hz (or 50 Hz) transforms the fundamental frequency into zero Hz and a much larger time step can be used in the EMTP solution. For those situations where the fundamental 60 Hz frequency is the most important frequency component, SFA results in considerable savings in solution time. The method offers a number of possibilities for its widespread application in power systems dynamic simulations [7], and for real-time operation and control applications.

Although “dynamic phasor” analysis has been studied for some time [8]-[13] the lack of clear understanding of its capabilities has challenged its widespread application. This paper aims at increasing this understanding. In addition to the time savings, phasor solutions facilitate the interface with dynamic controllers and PMU (Phasor Measurement Units) information.

The material presented in this paper is based on the work developed by our group at UBC [1]. In very interesting related work, [14]-[15], introduces the concept of a hybrid solution with an adaptive algorithm that combines an envelope solution based on an adaptable fundamental frequency with a standard EMTP solution with an adaptable time step. The SFA approach discussed here, however, uses a constant shifting frequency (60 Hz or 50 Hz) to first transfer the system from the original time domain into the SFA domain. A normal EMTP solution with a standard EMTP discretization rule and a constant time step is then used to solve the system in the SFA domain. The solution is then translated back into the original (unshifted) time domain.

In the paper, first the mathematical conceptual derivation of the SFA framework [1], [16] is presented. Subsequently, SFA models are derived for basic circuit components. The SFA algorithm is then implemented in an EMTP-based algorithm [6] with the use of MATLAB. A test case illustrates the main modelling concepts. Accuracy and stability of the method are assessed as a function of the size of the integration step and the discretization rule. It is expected that the relevance of this work will become more evident in future developments of general, continuous, and event-driven [1] computer simulation tools for application in power system dynamics studies, e.g., in supervisory control and energy centers in smart grids scenarios.

II. HILBERT TRANSFORM AND THE SFA – SHIFTED FREQUENCY ANALYSIS

In normal steady-state AC power system operation voltages and currents are sinusoidal waves of 60 Hz (or 50 Hz). To simplify the narration, 60 Hz will be assumed in this paper. During dynamic system disturbances, deviations from the 60 Hz base frequency can occur, resulting in a frequency band around the 60 Hz “carrier” component. In communication systems it is common to use a high frequency carrier frequency around which an information band is attached. After the signal has been transmitted, the information band is recovered by subtracting the carrier frequency. The Shifted Frequency Analysis (SFA) [3] closely resembles this approach.
The frequency spectrum of a real signal is symmetrical with respect to the vertical axis. In other words, a real signal is one that exhibits Hermitian symmetry between the negative and positive frequency components [16]. However, due to the implicit symmetry, the negative frequency components do not add any additional information about the system and the real analytic signal can be fully obtained from the positive frequencies alone. Despite this, it is still very advantageous to work with real signals (both positive and negative spectrums) due to their properties. In SFA, analytic signals are assumed to work with real signals (both positive and negative spectrums) frequencies alone. Despite this, it is still very advantageous to analytic signal can be fully obtained from the positive frequency components [16]). However, due to the implicit symmetry, the negative frequency components do not add any additional information about the system and the real analytic signal is defined as in (1):

\[ x_a(t) = x(t) + j\mathcal{H}[x(t)] \]

(1)

where \( \mathcal{H}[.\] is the Hilbert Transform. When the Hilbert Transform is applied to a real signal it becomes shifted by -90°, i.e., lagging by 90°. Formally the Hilbert Transform is defined as in (2):

\[ \mathcal{H}[x(t)] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(t)}{t-t'} dt \]

(2)

At this point we will have an analytic signal with a bandwidth frequency around the frequency of operation. The next step of the SFA-technique is to shift the analytic signal in order for it to have its bandpass frequency around zero (thus becoming a baseband signal), which makes it possible to use a larger integration time step. The SFA technique’s flowchart is presented in Fig. 1:

- **Step one** - Enter a real valued signal
- **Step two** - Generate the analytic signal
- **Step three** - Shift the analytic signal by \(-\omega_s\)

![Fig. 1. SFA flowchart.](image)

The SFA technique is illustrated through an example:

**Step one** – Enter a Real Valued function, such as in (3):

\[ x(t) = A \cos(\omega_s t + \theta) = \text{Re}\{A \cos(\omega_s t + \theta) + jA \sin(\omega_s t + \theta)\} = \text{Re}\{A e^{j(\omega_s t + \theta)}\} = \text{Re}\{A e^{j\omega_s t}\} \]

(3)

where \( \text{Re}\{ . \} \) is the real value of the complex signal. With the Fourier Transform applied to (3), one can obtain the frequency spectrum (with all frequencies contained in the system, or in this case, just the fundamental frequency), according to (4),

\[ \mathcal{F}\{A \cos(\omega_s t + \theta)\} = \frac{1}{2} \left[ A e^{j\theta} \delta(\omega - \omega_s) + A e^{-j\theta} \delta(\omega + \omega_s) \right] \]

(4)

where \( \delta(t) \) is the Dirac Delta function or unit impulse. This denotes the concept that a real signal has a symmetrical frequency spectrum.

**Step two** - obtain the analytic signal of the function entered in step one, by using (5) and (6):

\[ x_a(t) = x(t) + j\mathcal{F}[x(t)] = A \cos(\omega_s t + \theta) + jA \sin(\omega_s t + \theta) = Ae^{j\omega_s t} \]

(6)

By applying Fourier transform to (6), it results in (7):

\[ \mathcal{F}\{A \cos(\omega_s t + \theta) + jA \sin(\omega_s t + \theta)\} = \mathcal{F}\{A \cos(\omega_s t + \theta)\} + j\mathcal{F}[A \sin(\omega_s t + \theta)] = \left\{ \frac{1}{2} \left[ A e^{j\theta} \delta(\omega - \omega_s) + A e^{-j\theta} \delta(\omega + \omega_s) \right] \right\} + j \left\{ \frac{1}{2} \left[ A e^{j\theta} \delta(\omega - \omega_s) + A e^{-j\theta} \delta(\omega + \omega_s) \right] \right\} = A e^{j\omega_s} \delta(\omega - \omega_s) \]

(7)

**Step three** – translate or shift the system frequency spectrum by \(-\omega_s\), which means rotate the complex time domain signal, as in (8):

\[ x_{a,\text{shifted}}(t) = \left\{ A \cos(\omega_s t + \theta) + jA \sin(\omega_s t + \theta) \right\} e^{-j\omega_s t} = Ae^{j\omega_s t} e^{-j\omega_s t} = Ae^{j\theta} \]

(8)

By applying Fourier transform to (8), it is finally made clear, as shown in (9), that the single frequency \(\omega_s\) signal has been shifted to a frequency equal to zero (or DC):

\[ \mathcal{F}\{x_{a,\text{shifted}}(t)\} = Ae^{j\theta} \delta(0) = Ae^{j\theta} \]

(9)

Moreover, the signal \(x_{a,\text{shifted}}(t)\) expresses the concept of a “time varying phasor”, both in amplitude \(A(t)\) and phase angle \(\theta(t)\) (as long as their variations do not invalidate the transformations applied), which is also referred to as a Dynamic Phasor.

If the signal has multiple frequency components this concept can be expanded by using a selection of Fourier coefficients as described in [12]-[13]. Other time domain transformations are also discussed in [17].

It is possible to express the signal in the original “phase domain” (denoted with subscript “ph”) and the signal in the SFA domain, or modal domain (denoted with subscript “m”) by defining (10) and (11), from (3) and (8), respectively:

\[ x_{\text{ph}}(t) = A \cos(\omega_s t + \theta) = \text{Re}\{A \cos(\omega_s t + \theta) + jA \sin(\omega_s t + \theta)\} = \text{Re}\{A e^{j(\omega_s t + \theta)}\} = \text{Re}\{A e^{j\omega_s t}\} \]

(10)
\[ x_m(t) = A \cos(\omega_0 t + \theta) + jA \sin(\omega_0 t + \theta) \]
\[ e^{-j\omega t} = Ae^{j\theta} e^{-j\omega t} = Ae^{j\theta} \]

Therefore, one can define the following transformations,
\[ T = e^{j\omega t} \]
\[ T^{-1} = e^{-j\omega t} \]
which result in (14) and (15):
\[ x_m(t) = \{ x_{ph}(t) + j\mathcal{F}\{x_{ph}(t)\} \}, T^{-1} = [Ae^{j\theta} e^{j\omega t}], e^{-j\omega t} = Ae^{j\theta} \]
\[ x_{ph}(t) = Re\{Ae^{j\theta} e^{j\omega t}\} = Re\{x_m(t). T\} \]

Equations (14) and (15) can then be used to develop computer-based simulation models for power systems dynamic analysis in the SFA domain.

III. GRAPHICAL CONCEPTUAL DERIVATION OF THE SFA TECHNIQUE

The SFA technique is a modification of the basic EMTP algorithm to minimize the computational effort by displacing the fundamental frequency from 60 Hz to 0 Hz. Through the Hilbert transform, it is possible to transform the original “phase” system with frequency spectrum illustrated in Fig. 2 into a “modal” system, with the frequency spectrum illustrated in Figure 3 (the terms “phase” and “modal” are borrowed from transmission line modelling using transformation matrices).

Here the transformation is to apply a rotational transformation in the time domain (which means a translation or shifting in the frequency domain). By means of the SFA transformation, the frequency spectrum of Fig. 3 is transformed into the frequency spectrum of Fig. 4, as indicated in the detailed derivation of Section II. The idea with this rotation is to transform the original “around” 60 Hz components into new “around” 0 Hz (DC) components. The neighborhood of 60 Hz becomes the neighborhood of 0 Hz. The frequency range from 58 Hz to 62 Hz, for example, becomes the range from -2 Hz to 2 Hz. In a discrete time solution, the required time step size \( \Delta t \) is now the one needed to reproduce correctly up to 2 Hz. Denoting the deviation from the 60 [Hz] frequency as \( f_\Delta \), and using the rule that the maximum frequency in the simulation should be at least five times less than the Nyquist frequency, then:
\[ f_{\text{Nyq}} = \frac{1}{2\Delta t} \]
\[ \Delta t = \frac{1}{10f_\Delta} = \frac{1}{10 \times 2} = 50 [\text{ms}] \]

The crucial point of the SFA theory is that a finite \( \Delta t \) provides us in the simulation with a built-in lowpass filter that prevents frequencies higher than the Nyquist frequency (16) from being generated by the solution process. Therefore, there is an intrinsic low-pass anti-aliasing filtering in the SFA domain by the appropriate selection of the time step size. This assures that if our initial frequency band is within the box around zero Hertz indicated in Fig. 5, it will remain inside this box during the solution process, thus avoiding aliasing problems. In Fig. 5 the frequency spectrum of the shifted frequency signal has 0 [Hz] as the fundamental frequency and equal sidebands of width \( f_\Delta \). Notice that “the time response of a physically realizable system has to be causal and real valued”. Therefore, as defined in (15), in order to bring back the signal from the modal or SFA domain to the phase domain, first we rotate the modal domain signal by multiplying it by \( T = e^{j\omega t} \), and then we pick up its real part. That is, all we need to do to recover the real signal is to “drop” the imaginary part of the results [1].
IV. SFA Basic Modelling of Electrical Circuit Components

We discuss next how to implement the theory of Shifted Frequency Analysis (SFA) in a circuit analysis program such as the EMTP (or SPICE). As discussed before, one of the advantages of implementing this process in the EMTP solution is to be able to use very large time steps (\(\Delta t\)) and still get clean and correct simulation results in the neighborhood of the 60 Hz frequency. In EMTP modelling, each circuit component is represented by an equivalent discrete-time model that depends on the time step \(\Delta t\). Assuming that the sources are generated at time steps \(\Delta t\), the response of the circuit to this input will also be limited to frequencies below \(f_{Ny}\). In the EMTP circuit solution, the discretization step itself limits the frequency content of all variables in the circuit to be below the Nyquist frequency and there is no need for an additional lowpass antialiasing filter. The size of the discretization step provides the lowpass filter during the simulation.

A. Choosing the Step Size \(\Delta t_m\)

In EMTP simulation, the distortion of frequency components by the integration rule as we approach the Nyquist frequency can be used as a criterion to choose an adequate \(\Delta t\) for the simulation [6]. As a rule of thumb, using trapezoidal discretization in normal EMTP modelling, to limit the error to less than 3.3% in magnitude and zero degrees in phase angle, the maximum frequency we want to represent should be less than one fifth the Nyquist frequency.

For example, if we want to simulate frequencies between 58 [Hz] and 62 [Hz] in the phase domain (Fig. 3), this corresponds to \(f_3 = 2 [Hz]\) in the transformed domain (Fig. 4). The 3.3% rule requires \(f_{Ny} = \frac{1}{2\Delta t_m}\) to be five times higher than 2 [Hz], that is \(\Delta t_m = \frac{1}{10 \times 2} = 50 [ms]\). To simulate directly in phase coordinates, we would need a time step \(\Delta t_{ph} = \frac{1}{10 \times 62} = 1.61 [ms]\). With this criterion, the SFA solution would allow us to use a time step which is 30 times larger without loss of accuracy.

Detailed error analysis in Section V shows that SFA actually results in much higher accuracy than the expected from the normal EMTP solution. For example (Fig. 9), for \(\Delta f = 2Hz\), and \(\Delta t = 50ms\), the error with SFA and trapezoidal integration method is only 0.12% in magnitude and there is no error in phase. For an error of 3% in magnitude, we can actually increase the \(\Delta t\) to 183ms. The difference is that the SFA solution has two components, the 60Hz fundamental and the sidebands. The 60Hz component is simulated “exactly” and the numerical approximation only applies to the sideband frequencies.

B. Equivalent Circuits for R, L, C in the Shifted Frequency Domain

The equivalent circuits for the network components in the shifted frequency domain can be derived by first writing the component equations in the phase domain and then relating phase and modal variables according to the frequency shifting transformation as in (18)-(20):

\[
v_{ph}(t) = Re\{v_m(t).T\} \quad (18)
\]

\[
i_{ph}(t) = Re\{i_m(t).T\} \quad (19)
\]

\[
T = e^{j\omega_0 t} \quad (20)
\]

1) Resistance

For a resistance as in Fig. 6:

\[
v_{ph}(t) = R_i_{ph}(t) \quad (21)
\]

\[
Re\{v_m(t).T\} = Re\{R_i.m(t).T\} \quad (22)
\]

\[
v_m(t) = R_i.m(t).T.T^{-1} \quad (23)
\]

\[
v_m(t) = R_m.i_m(t) \quad (24)
\]

That is,

\[
R_m = R \quad (25)
\]

The modal resistance is the same as the phase resistance.

2) Inductance modeled with the trapezoidal integration rule in the discrete SFA domain

For an inductance (Fig. 7),

\[
v_{ph}(t) = L\frac{di_{ph}(t)}{dt} \quad (26)
\]

\[
Re\{v_m(t).T\} = Re\{L\frac{di_m(t).T}{dt}\} \quad (27)
\]

\[
v_m(t).T = L\left\{\frac{di_m(t)}{dt}.T + i_m(t).\frac{dT}{dt}\right\} = L\left\{\frac{di_m(t)}{dt}.T + j\omega_i.i_m(t).e^{j\omega_0 t}\right\} \quad (28)
\]

\[
v_m(t).T = L\left\{\frac{di_m(t)}{dt}.T + j\omega_i.i_m(t).T\right\} \quad (29)
\]

Cancelling out \(T\) in (29) gives,

\[
v_m(t) = L\frac{di_m(t)}{dt} + j\omega_i L_i_m(t) \quad (30)
\]

The equivalent circuits for the continuous-time inductance in the shifted frequency domain is shown in Fig. 7.
For an EMTP simulation in the SFA domain, (30) can be discretized using, for example, the trapezoidal integration rule. A direct way of doing this discretization is expressing the derivatives in terms of the derivative operator “s” \( s.x(t) = \frac{dx(t)}{dt} \) and then performing the trapezoidal discretization mapping (bilinear transformation) \( s = \frac{2}{\Delta t} \frac{1-z^{-1}}{1+z^{-1}} \), where \( z^{-1} \) is the delay-by-one operator, i.e., \( z^{-1}.x(t) = x(t - \Delta t) \). At this point in the derivation we will drop the subscript “m” in order to simplify the notation. We will pick it up again after we get the original forms. Equation (30), dropping the subscript is

\[
v(t) = L \frac{di(t)}{dt} + j \omega_o L \cdot i(t)
\]

Expressing (31) in terms of \( s \),

\[
V = L. s. I + j \omega_o L. I
\]

\[
V = L(s + j \omega_o)I
\]

\[
I = \frac{1}{s + j \omega_o} V
\]

(Equation 34 shows clearly the pole shifting by \( j \omega_o \) produced by the rotational transformation in time). We can now transform (33) to discrete time using the trapezoidal mapping,

\[
s = \frac{2}{\Delta t} \frac{1-z^{-1}}{1+z^{-1}}
\]

Introducing (35) in (33),

\[
v(t) = L \left( \frac{2}{\Delta t} \frac{1-z^{-1}}{1+z^{-1}} + j \omega_o \right) i(t)
\]

Defining

\[
R_L = \frac{2 \Delta t}{L}
\]

we can write:

\[
(1 + z^{-1})v(t) = R_L(1 - z^{-1})i(t) + j \omega_o L(1 + z^{-1})i(t) \tag{38}
\]

\[
v(t) + v(t - \Delta t) = [ R_L + j \omega_o L]i(t) - [ R_L - j \omega_o L]i(t - \Delta t) \tag{39}
\]

Defining in (39),

\[
Z_L = R_L + j \omega_o L
\]

as the “discretization” impedance, we can write

\[
v(t) + v(t - \Delta t) = Z_L.i(t) - Z_L^*i(t - \Delta t)
\]

\[
v(t) = Z_L.i(t) + [ -v(t - \Delta t) - Z_L^*i(t - \Delta t) ]
\]

\[
v(t) = Z_L.i(t) + e_{hl}(t)
\]

where \( e_{hl}(t) \) is a complex-valued history source for the solution at time \( t \). Bringing back subscript “m” to denote modal domain, the inductance branch is represented in the shifted frequency domain as

\[
v_m(t) = Z_L.i_m(t) + e_{hl_m}(t)
\]

\[
e_{hl_m}(t) = -v_m(t - \Delta t) - Z_L^*i_m(t - \Delta t)
\]

as shown in Figure 7-b). In EMTP simulation using nodal analysis, it is convenient not to have to evaluate the branch current in order to update the history. For a generic branch composed of an impedance \( Z \) and a voltage history source \( e_h(t) \), the branch voltage and branch current are related as

\[
v(t) = Z.i(t) + e_h(t)
\]

from which:

\[
i(t) = \frac{1}{Z} v(t) - \frac{1}{Z} e_h(t)
\]

\[
i(t - \Delta t) = \frac{1}{Z} v(t - \Delta t) - \frac{1}{Z} e_h(t - \Delta t)
\]

This value \( i(t - \Delta t) \) can be replaced in the history term of the branch equation. Applying this to (45),

\[
e_{hl_m}(t) = -v_m(t - \Delta t) - \frac{Z_L^*}{Z_L} v_m(t - \Delta t) + \frac{Z_L^*}{Z_L} e_{hl_m}(t - \Delta t)
\]

we finally get:

\[
e_{hl_m}(t) = -\frac{2R_L}{Z_L} v_m(t - \Delta t) + \frac{Z_L^*}{Z_L} e_{hl_m}(t - \Delta t)
\]

In (50), \( R_L = \frac{2 \Delta t}{L} \) and \( Z_L = \frac{2 \Delta t}{L} + j \omega_o L \).

The digital Thévenin equivalent circuit for the inductance with the trapezoidal integration rule in the discrete-time shifted frequency domain is shown in Fig. 7-b). However, it is convenient and necessary also to derive the digital Norton equivalent circuit for the inductance with the trapezoidal rule in the discrete-time shifted frequency domain, as illustrated in Fig. 7-c).

From (44) it is possible to express (51) as
\[ i_m(t) = \frac{1}{z_L} v_m(t) - \frac{1}{z_L} e_{hLm}(t) \] (51)

Defining:
\[ i_{hLm}(t) = \frac{1}{z_L} e_{hLm}(t) \] (52)

and using (45), results:
\[ i_{hLm}(t) = -\frac{v_m(t-\Delta t)}{z_L} - \frac{z_L^2}{z_L} i_m(t - \Delta t) \] (53)

Since
\[ i_m(t) = \frac{1}{z_L} v_m(t) - i_{hLm}(t) \] (54)
then
\[ i_m(t - \Delta t) = \frac{1}{z_L} v_m(t - \Delta t) - i_{hLm}(t - \Delta t) \] (55)

For the history source,
\[ i_{hLm}(t) = -\left(\frac{1}{z_L} + \frac{z_L^2}{z_L}\right) v_m(t - \Delta t) + \frac{z_L^2}{z_L} i_{hLm}(t - \Delta t) \] (56)

\[ i_{hLm}(t) = -\frac{2 z_L}{z_L} v_m(t - \Delta t) + \frac{z_L^2}{z_L} i_{hLm}(t - \Delta t) \] (57)

\[ i_{hLm}(t) = -2 \left(\frac{2 L}{\Delta t}\right) v_m(t - \Delta t) + \frac{1}{\Delta t} i_{hLm}(t - \Delta t) \] (58)

where: \( Y_L = \frac{1}{z_L} = \frac{1}{\Delta t + j \omega_0 L} \).

3) Capacitance modeled with the trapezoidal integration rule in the discrete SFA domain

From (60),
\[ i_{ph}(t) = C \frac{dv_{ph}(t)}{dt} \] (60)

a similar procedure can be used to derive the digital Thévenin equivalent capacitance model and the Norton digital equivalent capacitance model with the trapezoidal integration rule in the discrete-time SFA domain as illustrated in Fig. 8-b), and Fig. 8-c), respectively.

A general mapping directly from the Laplace domain to the discrete-time SFA domain with the trapezoidal integration rule can be obtained from (36) resulting in (61),
\[ s = \frac{2}{\Delta t} \frac{1 - e^{-z^{-1}}}{1 + z^{-1}} + j \omega_0 = \left(\frac{2}{\Delta t} + j \omega_0\right) - \left(\frac{2}{\Delta t} - j \omega_0\right) z^{-1} \] (61)

In Fig. 8-b):
\[ v_m(t) = Z_C \cdot i_m(t) + e_{hCm}(t) \] (62)
\[ e_{hCm}(t) = \frac{Z_C}{z_L} v_m(t - \Delta t) + Z_C \cdot i_m(t - \Delta t) \] (63)
\[ e_{hCm}(t) = \left(1 + \frac{z_L}{Z_C}\right) v_m(t - \Delta t) - e_{hCm}(t - \Delta t) \] (64)

In (62)-(64), \( Z_C = \frac{1}{\Delta t + j \omega_0 C} \), and \( R_C = \frac{\Delta t}{z_C} \).

In Fig. 8-c),
\[ i_m(t) = \frac{1}{z_C} v_m(t) - i_{hCm}(t) \] (65)
\[ i_{hCm}(t) = \left(\frac{1}{z_C} + \frac{1}{z_L}\right) v_m(t - \Delta t) - i_{hCm}(t) \] (66)
\[ i_{hCm}(t) = \frac{2}{r_L} v_m(t - \Delta t) - i_{hCm}(t - \Delta t) \] (67)
\[ i_{hCm}(t) = 2 \left(\frac{z_L}{\Delta t}\right) v_m(t - \Delta t) - i_{hCm}(t - \Delta t) \] (68)

In (65)-(68), \( R_C = \frac{\Delta t}{z_C} \), \( Z_C = \frac{1}{\Delta t + j \omega_0 C} \), and \( Y_C = \frac{1}{z_C} = \frac{2 C}{\Delta t} + j \omega_0 C \).

4) Sinusoidal Voltage Sources in the Shifted Frequency Domain

In shifted frequency domain solution, for the particular case when the applied source is a pure sinusoidal function (for example, in the study of power system 60 [Hz] dynamics), it results in
\[ e_{ph}(t) = A \cos(\omega_s t + \theta) = Re\{A e^{j(\omega_s t + \theta)}\} = Re\{A e^{j(\omega_s + \theta)}\} = \Re\{A e^{i \omega_s t} e^{j \theta}\} \] (69)

\[ e_m(t) = [e_{ph}(t) + j \beta e_{ph}(t)] e^{-j \omega_s t} = A e^{i \omega_s t} = A e^{j \theta + j \omega_s t} \] (70)

This result confirms our intuition that a sinusoidal 60 [Hz] source should become a DC source in the shifted frequency domain.
V. NUMERICAL ACCURACY ANALYSIS IN THE SFA DOMAIN

In order to perform a numerical accuracy analysis in the SFA domain, one can evaluate the model of an inductor as an integrator control block, where for a given input voltage the output will be the current [18]. Analogous analysis can also be done for a capacitor model in the SFA. The numerical accuracy depends on the integration method (Trapezoidal (71) or Backward Euler (72)), the frequency deviation ($\Delta f$) and the time step size ($\Delta t$). This is valid for any complexity of systems.

In this section, the equivalent admittance of an inductor modelled in the SFA domain as a function of the frequency is derived, considering a deviation around the fundamental frequency (60[Hz]), and compared with the ideal admittance of an inductor for the exact frequency value at the frequency $f$ ($Y_r(\Delta f)/Y(\Delta f)$). Fig. 9 and Fig. 10 present the error % in the magnitude and the error in degrees in the phase angle, respectively, for the Trapezoidal (73) and the Backward Euler (74) inductor model in the SFA domain, for a frequency deviation of $\pm 3\%$ and a time step size variation from 1 to 6 cycles of the fundamental frequency (60[Hz]).

$$Y_{\text{Tr}^{\text{p}}}(\Delta t) = \frac{1}{T^2} \left(1 - e^{-\Delta t} \right)$$ (71)

$$Y_{\text{BE}}(\Delta t) = \frac{1}{T} \left(1 - e^{-\Delta t} \right)$$ (72)

$$\text{Error}_{Y^{\text{Tr}^{\text{p}}}} = \frac{Y_{\text{Tr}^{\text{p}}}(\Delta t)}{Y(\Delta t)} - 1 \times 100$$ (73)

$$\text{Error}_{Y^{\text{BE}}} = \left( \frac{Y_{\text{BE}}(\Delta t)}{Y(\Delta t)} - 1 \right) \times 100$$ (74)

VI. TEST CASE

To illustrate the solution technique in the SFA domain, a simple case is presented here. This case is the solution for an R-L branch corresponding to a short-circuit in a network reduced to a Thévenin equivalent circuit with $R = 1\Omega$, $L = 10\text{mH}$; $V_{\text{source}} = (1.0V, 0.0\text{degrees})$. For the conventional EMTP solution: $\Delta t = 100\mu s$, $t_{\text{max}} = 100\text{ms}$.

Fig. 11 presents the instantaneous short-circuit current and its “dynamic phasor” magnitude and angle as a function of time. Fig. 12 presents the real and imaginary parts of the “dynamic phasor” for the short-circuit current during the simulation period. Fig. 13 shows the frequency centered spectrum of the real instantaneous short-circuit current (-60[Hz] and +60[Hz]) of the “dynamic phasor” (0[Hz]) with the SFA-EMTP simulation, which clearly illustrates the properties of Hilbert Transform and the frequency shifting.

Fig. 9. Error % in the magnitude of an inductor in the SFA domain for Trapezoidal and Backward Euler model, and a frequency deviation of $\mp 3\%$ with time step size variation from 1 to 6 cycles of the fundamental frequency (60[Hz]).

Fig. 10. Phase angle of Ye/Y [pu] for Trapezoidal and Backward Euler model, considering a frequency deviation of $\mp 3\%$ and time step size variation from 1 to 6 cycles of the fundamental frequency (60[Hz]).

Fig. 11. Instantaneous short-circuit current and its “dynamic phasor” magnitude and angle as a function of time.
This paper has presented a clear derivation of the Shifted Frequency Analysis (SFA) method based on the Hilbert transform. SFA can extend EMTP modelling to dynamic solutions when the power system oscillations remain close to the power system fundamental frequency (50Hz or 60Hz). The paper presents detailed error analysis that shows that, for narrow frequency bands around the fundamental frequency, SFA gives very accurate results for very large $\Delta t$’s when compared to normal EMTP solutions. The SFA solution provides the envelopes of the transient voltages and currents. If required, instantaneous values can be obtained directly by replacing the fundamental frequency with its time domain form. Since SFA is a “phasor” solution, it is particularly suited for real time implementation in conjunction with dynamic system controllers [19] in smart power grid environments. Work in implementing SFA in controllers for smart grids [20] distributed generation is currently under development.

VII. CONCLUSION

REFERENCES


