Volt/Var Optimization of Unbalanced Distribution Feeders via Mixed Integer Linear Programming

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Abstract— The paper presents a Mixed Integer Linear Programming (MILP) model for the solution of the three-phase volt/var optimization (VVO) of medium voltage unbalanced distribution feeders. The VVO of a distribution feeder is aimed at calculating the most efficient operating conditions by means of the scheduling of transformers equipped with an on-load tap changer and distributed reactive power resources (such as embedded generators and switchable capacitor banks). The proposed model allows the representation of feeders composed by three-phase, two-phase, and single-phase lines, by transformers with different winding connections, by unbalanced wye- and delta-connected loads, by three-phase and single phase capacitor banks and embedded generators. The accuracy of the results is verified by using IEEE Test Feeders.

Keywords— power distribution feeders, unbalanced networks, volt/var optimization, mixed integer linear programming.

I. INTRODUCTION

One of the active network management functions that modern distribution management systems are expected to provide is the so-called volt/var optimization (VVO) [1].

Different VVO definitions have been used in the technical literature. In this paper VVO problem refers to the determination of the set points of on-load tap changers (OLTCs) and switchable capacitor banks regulators with the objective of loss minimization, taking into account the usual operating constraints such as minimum and maximum voltage limits and ampacity (e.g. [2]). Modern and future distribution networks also require the optimization of the reactive power compensation capabilities provided by embedded generators (EGs) and by storage systems, with particular reference to those equipped by power electronic converters such as photovoltaic inverters (e.g. [3], [4] and references therein).

The typical differences between the VVO in distribution networks and the optimal power flow (OPF) problem in transmission systems are:

- usually VVO does not involve generator active power outputs as control variables, since EG outputs are mainly determined by the availability of energy resources and market/regulatory constraints;
- the main VVO control variables are integer (transformer taps and capacitor switching), whilst typical OPF control variables are continuous;
- voltage limits are usually more stringent than ampacity constraints;
- line and load unbalances appear in general more significant in distribution feeders than in transmission networks (this justifies the modelling effort in order to represent all the three phases of the system [5]).

Moreover, with respect to transmission system lines, distribution feeders are characterized by shorter length, higher ratio between resistance and reactance of the longitudinal impedance, and they are expected to transmit lower power flows. Therefore the influence of active power flows to voltage profile is not negligible and, usually, the maximum phase difference between the voltages at a bus in different operating conditions is limited to few degrees.

On the other hand VVO shares several aspects and solution approaches with OPF, so to justify the definition of distribution optimal power flow (DOPF) models (e.g. [6]). If the optimization horizon is larger than some tens of minutes, several uncertainties relevant to both renewable generators outputs and loads levels need to be taken into account (see, for example, [7]).

This paper aims at describing a Mixed Integer Linear Programming (MILP) model for the solution of the deterministic VVO of unbalanced distribution feeders. The configuration of the feeder is assumed to be known.

The motivation of using a MILP approach for VVO lays mainly on the availability of efficient solvers for this class of problems [8]. This approach has been recently investigated not only for distribution networks (e.g. [9]-[12]) but also for the solution of the classical OPF problem in transmission systems (e.g. [13], [14]).

The proposed model is a development of the one presented in [15] based on the representation of the distribution network with voltage – current relationships in Cartesian coordinates. The linear representation of loads and EGs is based on the assumption of limited deviations of bus voltage phasors with respect to the corresponding rated phasors. The MILP formulation presented in [15] assumes that the network is balanced. This paper presents a MILP model for a generic unbalanced distribution feeder, taking into account the different types of transformer winding connections, wye- and delta-connected loads, the presence of single-phase and three-phase EGs as well as the possibility to switch the capacitors on a per phase basis.

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As expected, the required computational effort is significantly increased with respect to the equivalent single phase model presented in [15].

The structure of the paper is the following. Section 2 describes the main characteristics of the implemented MILP model. Section 3 presents the complete model formalization. Section 4 illustrates the performances by the results obtained for some IEEE test feeders [16] with additional EGs.

II. PROPOSED LINEAR APPROACH

A. Objective Function

The considered deterministic static VVO aims at minimizing the active power consumption from the transmission network (and maximizing the active power injection), whilst minimizing the violation of the minimum and maximum voltage limits at all the buses:

\[
\min \left\{ \sum_{b=0}^{B} \left( V^p_{ib} + w_{min} \sum_{k=1}^{K} \chi^m_k + w_{max} \sum_{k=1}^{K} \chi^m_k \right) \right\}
\]

where \( B \) is the set of initial branches of the feeders connected to the substation; \( I_b \) indicates the current in branch \( b \); \( \chi^m_k \) and \( \chi^m_k \) indicate the violations of the minimum and maximum voltage bounds at node \( k \), respectively. \( w_{min} \) and \( w_{max} \) are the weights that penalize the corresponding violations.

As in [15], other than those shown in (1), the objective function includes additional terms that allow the penalized disconnection of loads and EGs if it is needed to find a feasible solution.

B. Branch Equations

The unbalanced per phase model is based on the representation of both lines and transformers with an equivalent network of uncoupled branches [17]. The equivalent representation for lines and transformers with various types of winding connections have been presented in several papers (e.g. [18]–[20]). Here below we review the models used to obtain the test results.

Tables I, II and III show the equations of the branches for the case of a three-phase line, a grounded wye - grounded wye transformer, A, B, C, and grounded node N identify the sending termination, whilst the corresponding lower case letters are used for the receiving termination.

| TABLE I. CONNECTION TABLE FOR THREE-PHASE LINES. |

<table>
<thead>
<tr>
<th>admittances</th>
<th>between nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>diag(Y) with Y = Z (^{-1})</td>
<td>A-a, B-b, C-c</td>
</tr>
<tr>
<td>0.5 ( \text{sum}(B(i,:)) )</td>
<td>A-N, B-N, C-N equal to a-n, b-n, c-n</td>
</tr>
<tr>
<td>Y(i, j)</td>
<td>A-b, A-c, B-a, B-b, C-a, C-b</td>
</tr>
<tr>
<td>( - (Y(i, j) + 0.5B(i, j)) )</td>
<td>A-B, B-C, A-C equal to a-b, b-c, a-c</td>
</tr>
</tbody>
</table>

The input data of the lines are the impedance and susceptance matrices (\( Z \) and \( B \)). Two-phase and single-phase lines are represented analogously to three-phase lines. The p.u. input data of the transformers are described by longitudinal resistance \( r_x \), longitudinal reactance \( x_x \), shunt admittance \( y_0 \) and ratio \( m \) (tap position at primary side is assumed fixed at nominal position).

| TABLE II. CONNECTION TABLE FOR GROUNDED WYE – GROUNDED WYE TRANSFORMER. |

<table>
<thead>
<tr>
<th>Admittances</th>
<th>between nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_m ) with ( y_m = \frac{1}{r + x} )</td>
<td>A-a, B-b, C-c</td>
</tr>
<tr>
<td>( y_m (1 - m) + y_0 )</td>
<td>A-N, B-N, C-N</td>
</tr>
<tr>
<td>( y_m (m - 1) )</td>
<td>a-n, b-n, c-n</td>
</tr>
</tbody>
</table>

| TABLE III. CONNECTION TABLE FOR DELTA – GROUNDED WYE TRANSFORMER. |

<table>
<thead>
<tr>
<th>Admittances</th>
<th>between nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_m / \sqrt{3} )</td>
<td>A-a, B-b, C-c</td>
</tr>
<tr>
<td>( y_m )</td>
<td>A-N, B-N, C-N</td>
</tr>
<tr>
<td>( -y_m / \sqrt{3} )</td>
<td>A-b, B-c, C-a</td>
</tr>
<tr>
<td>( y_m / \sqrt{3} )</td>
<td>a-n, b-n, c-n</td>
</tr>
</tbody>
</table>

Each branch from node \( h \) to \( k \) with admittance \( y_b \) is associated to two linear constraints

\[
V^p_{hb} - V^p_{kb} = r_b I^p_{hb} + x_b I^s_{hb} + \Delta V^p_{hb} = 0
\]

\[
V^s_{hb} - V^s_{kb} = r_b I^s_{hb} - I^s_{hb} + \Delta V^s_{hb} = 0
\]

where \( r_b + jx_b = y_b^{-1} \) that for the case of transformers equipped with an OLTC depends on tap position \( t \). Each tap position \( t \) is associated to auxiliary variables \( \Delta V^p_{ht} \) and \( \Delta V^s_{ht} \). If \( t \) is the selected tap position \( \Delta V^p_{ht} \) and \( \Delta V^s_{ht} \) are forced to be null by the following constraints, otherwise their value adapt to the voltage difference between the transformer terminals.

\[
\forall t = 1...n_t : \Delta V^p_{ht} + M^p u_{ht} \leq M^p \text{ and } - \Delta V^p_{ht} + M^p u_{ht} \leq M^p \Delta V^s_{ht} + M^s u_{ht} \leq M^s \text{ and } - \Delta V^s_{ht} + M^s u_{ht} \leq M^s \] (3)

\[
- I^p_{ht} u_{ht} \leq I^p_{thmax} \leq I^p_{htmax} \text{ and } - I^s_{ht} u_{ht} \leq I^s_{thmax} \leq I^s_{htmax} \] (4)

where \( u_{ht} \) is a binary variable associated to each tap position \( t \), \( n_t \) is the total number of tap positions and \( I^p_{thmax} \) is the maximum feasible value of \( I^p_{ht} \). \( M^p \) and \( M^s \) are tight but sufficiently large values that do not limit \( \Delta V^p_{ht} \) and \( \Delta V^s_{ht} \) when \( u_{ht} = 0 \).

For branches not associated to variable tap changers (e.g. lines and fixed ratio transformers), (2) simplifies being

\[
I^p_{ht} = I^p_{htmax} \text{, } I^s_{ht} = I^s_{htmax} \text{ and } \Delta V^p_{ht} = 0, \Delta V^s_{ht} = 0
\]
C. Load Model

Both wye- and delta-connected loads are represented. For the case of wye-connected loads, the voltage of the neutral is assumed to be 0 and the load current is assumed positive if drained from the phase node. If the load is delta connected, the positive load current is drained from one node (node $k$ in the following equations) and injected into a second node (node $h$).

Each load is represented by a weighted combination of constant power, constant current and constant impedance models:

$$P_{L,k} = p_{L,k} \left( \frac{V_k}{V_{0,k}} \right)^2 + b_{L,k} \frac{V_k}{V_{0,k}} + c_{L,k}$$

$$Q_{L,k} = Q_{L,k} \left( \frac{V_k}{V_{0,k}} \right)^2 + b_{L,k} \frac{V_k}{V_{0,k}} + c_{L,k}$$

(5)

where the coefficients satisfy the condition $a_{L,k} + b_{L,k} + c_{L,k} = a_{0,k} + b_{0,k} + c_{0,k} = 1$ and $P_{L,k}^0$, $Q_{L,k}^0$ are the real and reactive power consumed at node $k$ with rated voltage $V_{0,k}$ corresponding to a three-phase symmetrical system.

Cartesian coordinates $I_{L,k}^{re}$ and $I_{L,k}^{im}$ of the load current are defined by

$$I_{L,k}^{re} = I_{L,k}^{re} - \Delta I_{L,k}^{re} - \Delta I_{L,k}^{pr} = I_{k}^{p,pr} + I_{k}^{lm}$$

$$I_{L,k}^{im} = I_{L,k}^{im} - \Delta I_{L,k}^{lm} - \Delta I_{L,k}^{pr} = I_{k}^{p,lm} + I_{k}^{lm}$$

(6)

where $I_{k}^{p,pr}$, $I_{k}^{lm}$ correspond to $b_{L,k} P_{L,k}^0 + j c_{L,k} Q_{L,k}^0$ at $V_{0,k}$ and

$I_{L,k}^{re}$, $I_{L,k}^{im}$ correspond to $c_{L,k} P_{L,k}^0 + j b_{L,k} Q_{L,k}^0$ at $V_{0,k}$, whilst $I_{L,k}^{re}$, $I_{L,k}^{lm}$ represent the current drawn by a constant impedance load (with admittance $Y_{L,k} = G_{L,k} + j B_{L,k}$): 

$$I_{L,k}^{re} - G_{L,k} \left( V_k^{re} - V_{k0}^{re} \right) + B_{L,k} \left( V_k^{im} - V_{k0}^{im} \right) = 0$$

$$I_{L,k}^{im} - G_{L,k} \left( V_k^{im} - V_{k0}^{im} \right) + B_{L,k} \left( V_k^{re} - V_{k0}^{re} \right) = 0$$

(7)

$\Delta I_{L,k}^{re}$, $\Delta I_{L,k}^{lm}$, $\Delta I_{L,k}^{pr}$, $\Delta I_{L,k}^{pr}$ represent the variations of the current components relevant to the constant current model and constant power model due to the variation of the bus voltage.

The implemented linear representation is based on the assumption of small bus-voltage deviations with respect to $V_{0,k}$.

For $\Delta I_{L,k}^{re}$ and $\Delta I_{L,k}^{lm}$ we write

$$\Delta I_{L,k}^{re} = \left( \frac{I_{L,k}^{re}}{k_1} \right) \sin(\angle I_{L,k}^{re}) \left( \left( V_{k}^{im} - V_{k0}^{im} \right) \cos(\angle V_k^{re} + \angle k_2) - \left( V_{k0}^{im} - V_{k0}^{im} \right) \sin(\angle V_k^{re} + \angle k_2) \right) = 0$$

$$\Delta I_{L,k}^{lm} = \left( \frac{I_{L,k}^{lm}}{k_2} \right) \sin(\angle I_{L,k}^{lm}) \left( \left( V_{k}^{re} - V_{k0}^{re} \right) \cos(\angle V_k^{im} + \angle k_2) + \left( V_{k0}^{re} - V_{k0}^{re} \right) \sin(\angle V_k^{im} + \angle k_2) \right) = 0$$

(8)

where $k_1 = \sqrt{3}$ and $k_2 = \pi / 6$ if the load is delta connected, otherwise they are 1 and 0 respectively.

Analogously, for $\Delta I_{L,k}^{pr}$ and $\Delta I_{L,k}^{lm}$:

$$\Delta I_{L,k}^{pr} = \left( \frac{I_{L,k}^{pr}}{k_3} \right) \sin(\angle I_{L,k}^{pr}) \left( \left( V_{k}^{im} - V_{k0}^{im} \right) \cos(\angle V_k^{pr} + \angle k_2) + \left( V_{k0}^{im} - V_{k0}^{im} \right) \sin(\angle V_k^{pr} + \angle k_2) \right) + In_{L,k}^{pr}\left( \left( V_{k}^{re} - V_{k0}^{re} \right) \sin(\angle V_k^{pr} + \angle k_2) + \left( V_{k0}^{re} - V_{k0}^{re} \right) \cos(\angle V_k^{pr} + \angle k_2) \right) +$$

$$+ In_{L,k}^{lm}\left( \left( V_{k}^{re} - V_{k0}^{re} \right) \sin(\angle V_k^{lm} + \angle k_2) + \left( V_{k0}^{re} - V_{k0}^{re} \right) \cos(\angle V_k^{lm} + \angle k_2) \right) = 0$$

(9)

D. Embedded Generators

Whilst single phase generators are represented as PQ loads (with opposite power sign), three phase generators are represented as three symmetrical current sources in parallel with admittance matrix $Y_{G}$.[19]

In order to include the choice of different reactive power outputs levels in the MILP model, $\eta_{G,k}$ discrete compensation levels are represented. Each level $g$ corresponds to reactive power output $Q_{G,k}^{pg}$ or power factor $\theta_{G,k}^{pg}$, whilst active output $P_{G,k}^{pg}$ is assumed fixed. In [12] a different MILP formulation based on the use of McCormick’s envelopes is proposed.

For three-phase EGs at bus $k$ (with nodes $k, k_1$ and $k_2$) we write

$$\Gamma_{G,k}^{pr} = \sum_{g=1}^{n_{G,k}} \eta_{G,k}^{pg} \Re\{Y_{G,k} V_{G,k}^{pg}\} = 0$$

$$\Gamma_{G,k}^{lm} = \sum_{g=1}^{n_{G,k}} \eta_{G,k}^{pg} \Im\{Y_{G,k} V_{G,k}^{pg}\} = 0$$

(10)

where $V_{G,k}$ is the triplet of voltage phasors at bus $k$; $\Gamma_{G,k}^{pr}$ and $\Gamma_{G,k}^{lm}$ are the triplets of the real and imaginary coordinates of the currents phasors injected by the EG in each phase of bus $k$; $\eta_{G,k}^{pg}$ and $\eta_{G,k}^{lm}$ are the triplets corresponding to the reactive power compensation level $g$ (only those corresponding to the selected level are not null due to the following constraints).

$$\forall g = 1...n_{G,k}:$$

$$\Delta I_{k_g}^{pr} + M_{k_g}^{pr} u_{k_g} \leq M_{k_g}^{pr}$$

and

$$\Delta I_{k_g}^{lm} + M_{k_g}^{lm} u_{k_g} \leq M_{k_g}^{lm}$$

(11)

$$I_{k_g}^{pr} + M_{k_g}^{pr} u_{k_g} \leq M_{k_g}^{pr}$$

and

$$I_{k_g}^{lm} + M_{k_g}^{lm} u_{k_g} \leq M_{k_g}^{lm}$$

(12)

$$I_{g_{max}}^{pr} u_{k_g} \leq I_{g_{max}}^{pr} \leq I_{g_{max}}^{pr} u_{k_g}$$

(13)

with $\sum_{g=1}^{n_{G,k}} \eta_{G,k}^{pg} = 1$. $u_{k_e}$ is a binary variable associated to each level $g$. $M_{k_g}^{pr}$ and $M_{k_g}^{lm}$ are tight values that do not limit $\Delta I_{k_g}^{pr}$ and $\Delta I_{k_g}^{lm}$ when $u_{k_g} = 0$.

For phase $a$ of bus $k$ (indicated as $k_a$):

$$I_{G,k_a}^{pr} - \Delta I_{L,k}^{re} + I_{G,k_a}^{pr} = I_{G,k_a}^{pr}$$

$$I_{G,k_a}^{lm} - \Delta I_{L,k}^{lm} + I_{G,k_a}^{lm} = I_{G,k_a}^{lm}$$

(14)
with $\Delta P_{C^{\text{re}}}^{k}$ and $\Delta P_{C^{\text{im}}}^{k}$ defined by, analogously to (9),

$$
\Delta P_{C^{\text{re}}}^{k} = \Re(\Delta V_{C^{\text{re}}}^{k}) + \text{Im}(\Delta V_{C^{\text{im}}}^{k})^2 + \text{Im}(\Delta V_{C^{\text{im}}}^{k})^2 + \text{Re}(\Delta V_{C^{\text{re}}}^{k})^2 + \text{Re}(\Delta V_{C^{\text{im}}}^{k})^2 - \text{Im}(\Delta V_{C^{\text{re}}}^{k})^2 - \text{Re}(\Delta V_{C^{\text{im}}}^{k})^2
$$

(15)

$$
\Delta P_{C^{\text{im}}}^{k} = \Delta V_{C^{\text{re}}}^{k} \cos(\angle V_{C^{\text{re}}}^{k}) + \Delta V_{C^{\text{im}}}^{k} \sin(\angle V_{C^{\text{re}}}^{k}) + \text{Im}(\Delta V_{C^{\text{re}}}^{k})^2 + \text{Re}(\Delta V_{C^{\text{im}}}^{k})^2 - \text{Im}(\Delta V_{C^{\text{re}}}^{k})^2 - \text{Re}(\Delta V_{C^{\text{im}}}^{k})^2
$$

where $\Delta V_{C^{\text{re}}}^{k}$ and $\Delta V_{C^{\text{im}}}^{k}$ indicate the positive sequence component of $\Delta V_k$; $\Delta V_{C^{\text{re}}}^{k}$ and $\Delta V_{C^{\text{im}}}^{k}$ correspond to the active and reactive power injections at compensation level $g$ and at rated voltage $V_{g,k}$ calculated by

$$
\text{Im}(\Delta V_{C^{\text{re}}}^{k}) = \frac{P_{C^{\text{re}}}^{k}(k) - jQ_{C^{\text{im}}}^{k}(k)}{V_{g,k}^2}
$$

(16)

being $V_{g,k}$ the positive sequence phasor of $V_{g,k}$ and * the conjugate (transpose).

### E. Capacitor Banks

A binary variable $u_{n,k}^{C}$ is associated to each switch $n,w$ of a bank with $n, c$ capacitors connected to node $k$. Current $I_{C,k}^{\text{re}} + jI_{C,k}^{\text{im}}$ derived by the capacitor is provided by the summation of the currents associated with each switch position:

$$
I_{C,k}^{\text{re}} = \sum_{n=1}^{N} I_{C,n,k}^{\text{re}} = 0 \quad \text{and} \quad I_{C,k}^{\text{im}} = \sum_{n=1}^{N} I_{C,n,k}^{\text{im}} = 0
$$

(17)

At least one current $I_{C,n,k}^{\text{re}} + jI_{C,n,k}^{\text{im}}$ is not null due to the following constraints.

$$
\forall n,w = 1...n_{C,k}:

I_{C,n,k}^{\text{re}} - \sum_{w=1}^{N} I_{C,w,k}^{\text{re}} = 0 \quad \text{and} \quad I_{C,n,k}^{\text{im}} - \sum_{w=1}^{N} I_{C,w,k}^{\text{im}} = 0
$$

(18)

The minimum and maximum node voltage and maximum branch current constraints are represented through the description of the feasible region by means of polygons, as described in [15]. The MILP model also includes a minimum power factor constraint enforced for each phase at the slack bus.

### III. TEST RESULTS

The model described in the previous section has been implemented in Matlab R2012a and solved by using IBM ILOG CPLEX V12.5 MIP on a computer with two 3.07 GHz Intel six-core processors and 48 GB of RAM, running 64-bit Windows.

Numerical tests have been carried out for the following networks adapted from IEEE test feeders:

* TS1 IEEE 13 Node Test Feeder, with two additional EGs, namely a single-phase unit at bus 646 and a three-phase unit at bus 680 (Fig. 1);
* TS2 IEEE 34 Node Test Feeder with a three-phase EG unit at bus 888 (Fig. 2);
* TS3 IEEE 123 Node Test Feeder, with a three-phase EG unit at node 56 and a single-phase EG unit at node 104 (Fig. 3).

The results show the effectiveness of the proposed model in handling active and reactive power flow constraints, as well as voltage and branch current limits. The MILP model is able to accurately compute the optimal operation of the distribution system, ensuring both technical and economic feasibility.

The simulations were carried out with a MATLAB environment on a computer with double-core processors and 16 GB of RAM, running 64-bit Windows.

### G. Bus Equations

At each phase of the slack bus, voltage is set in both magnitude and phase (three-phase symmetrical system). In each node $k$ that does not belong to the slack bus the equilibrium of the currents is forced:

$$
\sum_{b} I_{b}^{\text{re}} - \sum_{b} I_{b}^{\text{im}} + \sum_{b} I_{b}^{\text{sh},\text{re}} + I_{k}^{\text{re}} + I_{k}^{\text{im}} - I_{k}^{\text{sh},\text{re}} - I_{k}^{\text{sh},\text{im}} = 0
$$

where $b$ is the set of lines $b$ connected to node $k$ (being $b_{k}$ the set to the lines leaving $k$ and $b_{k}$ the set of those entering in $k$). If $k$ is connected to the secondary side of a distribution voltage regulator represented with the simplified model then $I_{k}^{\text{re}}$ and $I_{k}^{\text{im}}$ are replaced by $I_{k}^{\text{re}}$ and $I_{k}^{\text{im}}$ of (23).

$$
V_{k}^{\text{re}} - m_{k} V_{k}^{\text{im}} + \Delta V_{k}^{\text{re}} = 0 \quad \text{and} \quad V_{k}^{\text{im}} - m_{k} V_{k}^{\text{im}} + \Delta V_{k}^{\text{im}} = 0
$$

(22)

being $h$ and $k$ the nodes of the same phase at the primary and secondary side respectively, $m_{k}$ is the ratio corresponding to tap position $k$.

Constraints (3) and (4) still apply also for this simplified model where $I_{k}$ refers to the current at the primary side of each phase. In order to calculate the Cartesian coordinates of the corresponding current phasor at the secondary side ($I_{k}$), the following constraints are added:

$$
I_{k}^{\text{re}} - \sum_{b} m_{b} I_{b}^{\text{re}} = 0 \quad \text{and} \quad I_{k}^{\text{im}} - \sum_{b} m_{b} I_{b}^{\text{im}} = 0
$$

(23)

The model described in the previous section has been implemented in Matlab R2012a and solved by using IBM ILOG CPLEX V12.5 MIP on a computer with two 3.07 GHz Intel six-core processors and 48 GB of RAM, running 64-bit Windows.

Numerical tests have been carried out for the following networks adapted from IEEE test feeders:

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than 50 kvar, respectively.

equal step increments if the maximum power is equal or greater
independently switchable on a per phase basis with one or two
transformers are neglected. The capacitors banks are

Fig. 2. Test system TS2. Figure adapted from [16].

branch limits are set large enough so they are never binding the
limit on each phase at the substation is 0.8, whilst the current
limits are 0.95 pu and 1.05 pu and the minimum power factor
neglected. Distributed loads are assumed to be concentrated at

Fig. 3. Test system TS3. Figure adapted from [16].

In all the models, the magnetizing branches of the
transformers are neglected. The capacitors banks are
independently switchable on a per phase basis with one or two
equal step increments if the maximum power is equal or greater
than 50 kvar, respectively. Y C of the three-phase EG models is
neglected. Distributed loads are assumed to be concentrated at
the remote line termination. The lower and upper bus voltage
limits are 0.95 pu and 1.05 pu and the minimum power factor
limit on each phase at the substation is 0.8, whilst the current
branch limits are set large enough so they are never binding the
solution.

The following subsections present the results obtained for
the considered test feeders by means of tables that report the
optimal values of the control variables calculated by the MILP
model, the number of variables (total number and number of
binary variables), the number of constraints, the final relative
objective gap (i.e. the difference between the best solution and
its best lower bound divided by the best solution value) and the
computer time spent to obtain the solution. In parenthesis,
the tables indicate the percentage differences between the MILP
results and those obtained by the corresponding power flow
(PF) calculation (the EMTP-rv 2.5 software is used). These
differences provide an indication of the influence of the adopted linear approximation on the accuracy of the results
obtained by using the MILP model.

A. TS1

The slack bus is at the primary side of the 5 MVA D-Y
substation transformer with voltage equal to V s=115 kV. This
transformer is equipped with an OLTC with ±12 tap
increments of 1.25% with rated ratio equal to 115/4.16 kV. The
fixed ratio of the Y-Y transformer feeding bus 634 is
4.16/0.48 kV. One-phase switched capacitor bank with
maximum power of 100 kvar is connected at node 611c and a
three-phase bank at bus 675 (maximum power of 200 kvar at
each phase). The two additional EGs are: a 400 kW single-
phase unit at node 646b and a 1200 kW three-phase unit at bus
680. The EGs have the possibility to control the reactive output
in ±12 discrete levels with a minim power factor limit of 0.7.

Table IV shows the results obtained with and without the
presence of EGs for two different load levels: normally loaded
(i.e. as in IEEE Test Feeder data) and light-loaded (i.e.
obtained by multiplying each load demand by 0.5).

Without EGs, the system without regulation (i.e. with all
capacitors disconnected and OLTC in the 0 tap position) has
- at normal load: losses = 211.01 kW, minimum bus voltage
= 0.804 pu (node 611c) and min power factor at the
substation = 0.71 (phase b);
- at light load: losses = 45.9 kW, min voltage = 0.911 pu
(node 611c) and min power factor = 0.77 (phase b).

With the action of the OLTC and capacitor banks, the
calculated operating condition slightly violates the ±5% voltage
limits as well as the minimum power factor constraint (the
relevant penalties are added to the objective function value)
whilst all the constraints are met at light load. Losses are
reduced by around a quarter in both load conditions.

With the EGs, the system without regulation (i.e. with all
capacitors disconnected, OLTC in the 0 tap position and EG at
unity power factor) has
- at normal load: losses = 129.7 kW, min voltage = 0.817 pu
(node 611c), max voltage = 0.982 pu (node 680b) and min
power factor at the substation = 0.35 (phase b);
- at light load: losses = 29.9 kW, min voltage = 0.92 pu
(node 611c), max voltage = 1.00 pu (646b) and min power factor
= 0.18 (phase b).

Losses are significantly reduced in both load conditions
thanks to the action of the OLTC, of capacitor banks and of EG
regulation. At normal load, the power factor on phase b of
slack bus is lower than 0.8 and the maximum voltage limit is
exceeded at node 680b, whilst at light load the power factor
constraint is violated on phase b and c (the relevant penalties
are added to the objective function value).
substation transformer with voltage equal to $b$ and phase $c$, respectively.

386, 397 kW and 380, 456, 375 kvar at normal load whilst 433,

obtained on each phase of the three-phase EG at bus 680: 416,

network imbalance different active and reactive powers are

three-phase unit with the possibility to control the reactive

phase), respectively. The additional EG at bus 888 is a 500 kW

kvar per phase) and to bus 848 (with maximum 150 kvar per

increments of 1.25% at the secondary side. Two three-phase

transformer is assumed at fixed ratio equal to 2.56. Moreover

there are two 4.16/4.16 kV distribution voltage regulators

connected before nodes 850 and 832, each enabling ±12 tap

increments of 1.25% at the secondary side. Two three-phase
capacitor banks are connected to bus 844 (with maximum 100 kvar per phase) and to bus 848 (with maximum 150 kvar per phase), respectively. The additional EG at bus 888 is a 500 kW three-phase unit with the possibility to control the reactive

output in ±6 discrete levels with a minimum power factor limit of 0.7.

Table V shows the results obtained without and with the EG.

Although the internal $Y_G$ matrix is neglected, due to the network imbalance different active and reactive powers are obtained on each phase of the three-phase EG at bus 680: 416, 386, 397 kW and 380, 456, 375 kvar at normal load whilst 433, 368, 394 kW and 400, 433, 373 at light load on phase $a$, phase $b$ and phase $c$, respectively.

### B. TS2

The slack bus is at the primary side of the 2.5 MVA D-Y

substation transformer with voltage equal to $V_e=69$ kV. The

transformer is assumed at fixed ratio equal to 2.56. Moreover

there are two 4.16/4.16 kV distribution voltage regulators

connected before nodes 850 and 832, each enabling ±12 tap

increments of 1.25% at the secondary side. Two three-phase
capacitor banks are connected to bus 844 (with maximum 100 kvar per phase) and to bus 848 (with maximum 150 kvar per phase), respectively. The additional EG at bus 888 is a 500 kW three-phase unit with the possibility to control the reactive

output in ±6 discrete levels with a minimum power factor limit of 0.7.

Table V shows the results obtained without and with the EG.

### TABLE IV. MILP SOLUTIONS FOR TS1 (IN PARENTHESIS THE PERCENTAGE DEVIATION WITH RESPECT TO THE PF RESULTS).

<table>
<thead>
<tr>
<th>Cap. steps at nodes 611, 675abc</th>
<th>without EG</th>
<th>with EG</th>
</tr>
</thead>
<tbody>
<tr>
<td>normal load</td>
<td>2,2,0,2</td>
<td>2,2,0,2</td>
</tr>
<tr>
<td>light load</td>
<td>2,2,1,2</td>
<td>2,2,0,2</td>
</tr>
<tr>
<td>losses</td>
<td>161.79 kW</td>
<td>70.22 kW</td>
</tr>
<tr>
<td>min voltage elsewhere than slack bus</td>
<td>9.536 pu</td>
<td>9.514 pu</td>
</tr>
<tr>
<td>max voltage elsewhere than slack bus</td>
<td>1.056 pu</td>
<td>1.0552 pu</td>
</tr>
<tr>
<td>tot. active load</td>
<td>3.507 MW</td>
<td>3.805 MW</td>
</tr>
<tr>
<td>tot. reactive load</td>
<td>2.096 Mvar</td>
<td>2.095 Mvar</td>
</tr>
<tr>
<td>EG tot. active injection</td>
<td>1.605 Mvar</td>
<td>1.603 Mvar</td>
</tr>
<tr>
<td>EG tot. reactive injection</td>
<td>1.230 Mvar</td>
<td>924 kvar</td>
</tr>
<tr>
<td>$P$ at slack bus</td>
<td>3.668 MW</td>
<td>1.952 MW</td>
</tr>
<tr>
<td>$Q$ at slack bus</td>
<td>2.298 Mvar</td>
<td>647 kvar</td>
</tr>
<tr>
<td>$pf$ at slack bus (abc)</td>
<td>0.85, 0.89</td>
<td>0.96, 0.74</td>
</tr>
<tr>
<td>no. of variables</td>
<td>1520</td>
<td>2182</td>
</tr>
<tr>
<td>obj. function</td>
<td>5.13 10^3</td>
<td>4.23 10^3</td>
</tr>
<tr>
<td>relative obj. gap</td>
<td>0</td>
<td>9.41 10^4</td>
</tr>
<tr>
<td>CPU time</td>
<td>2.05 s</td>
<td>37.59 s</td>
</tr>
</tbody>
</table>

### TABLE V. MILP SOLUTIONS FOR TS2 AND TS3 (IN PARENTHESIS THE PERCENTAGE DEVIATION WITH RESPECT TO THE PF RESULTS).

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<tr>
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</tr>
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<tbody>
<tr>
<td>normal load</td>
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<td>light load</td>
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### TABLE V. MILP SOLUTIONS FOR TS2 AND TS3 (IN PARENTHESIS THE PERCENTAGE DEVIATION WITH RESPECT TO THE PF RESULTS).

<table>
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</table>
With the EG and without regulation (i.e. with all capacitors disconnected, OLTC in the 0 tap position and EG at unity power factor) the system has losses = 188.4 kW, minimum bus voltage = 0.834 pu (node 890a), total load equal to 1.588 MW, min power factor at the substation = 0.77 (phase c). With the action of the voltage regulators and capacitor banks, the calculated operating condition only slightly violates the +5% voltage limit at node 800e. The load rises by 13% whilst the power loss decreases less than 1%.

C. TS3

The slack bus voltage is equal to \( V_s = 115 \) kV. The 5 MVA D-Y OLTC substation transformer has rated ratio equal to 115/4.16 kV with ±12 tap increments of 1.25%. Moreover there is a three-phase 4.16/4.16 kV distribution voltage regulator connected before node 67 with ±12 tap increments of 1.25%. The other two distribution voltage regulators and the 4.16/0.48 kV transformer shown in Fig. 3 (connected between buses 25-26, 9-14, 61-610, respectively) are considered at fixed ratio. A three-phase capacitor bank is connected to bus 83 (with maximum 200 kvar per phase). Three single-phase capacitor banks are connected to node 88a, 90b, and 92c (with maximum 50 kvar). The two additional EGs are a 1.2 MW three-phase unit at bus 56 and a 400 kW single-phase unit at node 104c both allowing ±6 discrete reactive power levels with a minim power factor limit of 0.85.

Also the results obtained for TS3 without and with the EG are shown in Table V. Without EGs, the system without regulation (i.e. with all capacitors disconnected and OLTCs in the 0 tap position) has losses = 152.7 kW, minimum bus voltage = 0.85 pu (node 114a), total load equal to 3.269 MW, min power factor at the substation = 0.76 (phase b). With the action of the voltage regulators and capacitor banks, the calculated operating condition meets all the constraints. Although a roughly 5% load increase, power loss decreases by 9%. With the EGs and without regulation (i.e. with all capacitors disconnected, OLTC in the 0 tap position and EGs at unity power factor) the system has losses = 102.46 kW, minimum bus voltage = 0.85 pu (node 114a), total load equal to 3.3 MW, min power factor at the substation = 0.53 (phase c). With the action of the voltage regulators and capacitor banks, the calculated operating condition meets all the constraints, load increases by 4.7% and the power loss decreases by nearly 30%.

IV. CONCLUSIONS

The paper presents a MILP model for the solution of the VVO in unbalanced three-phase distribution feeders. The model includes the specific characteristics of the components and takes the usual operating constraints into account.

The method has been applied to three different IEEE test feeders with OLTC transformers, distribution voltage regulators, switchable capacitor banks, and EGs. The comparison between the obtained results and those provided by the three-phase PF calculations (for the optimal configuration) shows that the achieved accuracy is adequate.

As expected, the computational effort significantly increases with respect to a previously presented MILP model that assumes a balanced network. However, with acceptable CPU time, accurate results are obtained also for medium size feeders.

REFERENCES


