Abstract—For the sub-hourly real-time optimization of a hydropower cascade, it is important to model in detail the three factors which most heavily influence hydropower generation: reservoir elevation, tailrace elevation, and turbine-generator efficiency. The method proposed in this paper uses three-dimensional piecewise linearization to accurately approximate the nonlinear and non-convex hydropower production function as a function of hydraulic head and turbine discharge. We show how this piecewise linear approximation can be implemented in a model predictive control framework with promising results. We present a case study of this model applied to seven dams along the Columbia River in the United States, with a particular focus on simulating the integration of variable wind generation. The primary contribution of this paper is to introduce a new method to model power generation from hydropower facilities and to simulate and demonstrate its efficacy on a real-world system.

Keywords—Hydroelectric power, power generation scheduling, optimization methods, quadratic programming

I. INTRODUCTION

Over the past decade, there has been a sustained push to supplement and replace conventional thermal generation with renewables, especially wind power. However, renewable electricity generation suffers from variations in output power resulting from the inherent stochasticity of weather patterns. When renewable penetration is low, this variability can be smoothed via existing regulation and load following capability. However, as more wind power is added to the grid, the variability from non-dispatchable renewable generation quickly dominates the variability from electricity demand. Our inability to consistently and accurately predict these deviations means that adding a significant amount of these resources to the grid stresses the power system’s ability to remain balanced [1].

Fortunately, with fast ramping capability, sizeable storage volumes, and inexpensive fuel, hydropower is ideal for smoothing out these power fluctuations. Hydropower also has the requisite levels of grid penetration to make it an economically viable option. Of course, balancing the variability from non-dispatchable renewables like wind requires an operational approach that fully leverages the flexibility a hydropower cascade can provide. There are examples in the literature demonstrating the benefits of operating hydropower and wind power symbiotically to increase economic profit [2], mitigate transmission congestion [3], and reduce wind curtailment [4].

The objective of this paper is to devise a real-time optimization scheme for a hydropower cascade that considers the dynamics of the system and accurately models the hydropower production function. This will allow us to simulate the sub-hourly real-time dispatch of a hydropower system and evaluate its performance under different wind and load scenarios. Despite being formulated as a quadratic program, the proposed modeling approach still matches the ability of mixed-integer or non-linear models to fully characterize the hydropower production function [5], [6]. This results in faster simulation times and the capability to do large-scale studies on the effect of wind generation on hydropower operations.

The basic system model is devised in Section II. In Section III, we describe how we model dynamic tailrace elevation. We devise a three-dimensional piecewise linear approximation to the hydropower production function in Section IV. We formulate the optimization problem in Section V and present a case study in Section VI. Section VII concludes the paper.

II. MODEL PREDICTIVE CONTROL

In this paper, we employ a control technique known as Model Predictive Control (MPC) [7]. In MPC, a state-space model is used to predict the reaction of a system to a set of control inputs. The control sequence that delivers the best performance and keeps the system within well-defined constraints is chosen as the optimal control sequence. The first step of that control sequence is then applied, and the system is observed at the next time step. The process then begins again, repeating itself indefinitely. This method of receding horizon control is resilient to unexpected disturbances, has explicit constraints on control and system variables, and incorporates forecasts of future inputs.

Linear, time-discrete MPC models have the general form

\[
\begin{align*}
    x(k+1) &= Ax(k) + Bu(k) \quad \text{for } k = 0, \ldots, K - 1 \\
    y(k) &= Cx(k) \quad \text{for } k = 1, \ldots, K
\end{align*}
\]

with constraints

\[
\begin{align*}
    u^\text{min} &\leq u(k) \leq u^\text{max} \quad \text{for } k = 0, \ldots, K - 1 \\
    x^\text{min} &\leq x(k) \leq x^\text{max} \quad \text{for } k = 1, \ldots, K \\
    y^\text{min} &\leq y(k) \leq y^\text{max} \quad \text{for } k = 1, \ldots, K
\end{align*}
\]

where \( u(k) \) is the vector of control variables, \( x(k) \) is the vector of state variables, and \( y(k) \) is the vector of observed variables. The \( A \) and \( B \) matrices describe the relationship between the control inputs, current system state, and future system state. Similarly, the \( C \) matrix is the relationship between the system
state $x(k)$ and the observed system state $y(k)$. $K$ is the discrete time-horizon over which this system is optimized. Inequality constraints on state, control, and observed variables are incorporated as explicit constraints (2) into the MPC model. Typically, the objective function for a linear, time-discrete MPC optimization problem is of the form

$$\min_{x,u} \sum_{k=0}^{K-1} \left[ x(k)^T Q x(k) + u(k)^T R u(k) + \Psi_j(\delta q_j) \right]$$

where the weight matrices $Q$ and $R$ are chosen to minimize particular objectives [7].

For a hydropower cascade with no branching, we use the MPC framework (1) to model system hydraulics as

$$x_j(k+1) = x_j(k) + \frac{t_j}{\Psi_j} w_j(k)$$

$$+ \frac{t_j}{\Psi_j} \left( q_{j-1}(k) - q_j(k) \right)$$

$$+ \frac{t_j}{\Psi_j} \left( s_{j-1}(k) - s_j(k) \right)$$

(4)

where natural inflow into the reservoir behind dam $j$ is denoted by $w_j(k)$; turbine discharge and spill through dam $j$ is denoted by $q_j(k)$ and $s_j(k)$, respectively; and the water level behind dam $j$ is denoted by $x_j(k)$. $\Psi_j$ is the surface area of the reservoir behind dam $j$. The model is discretized by $t_j$, the length of the optimization interval. There are a total of $J$ dams in the cascade. Additionally, since the state variable cannot change instantaneously, $x_j(0)$ is a fixed value reflecting the current reservoir elevation.

Note that the hydraulic model above does not utilize any output states $y(k)$. Thus, the complete system model is given by (1a) where the state vector $x(k)$ is the reservoir level behind each dam and the elements in the control vector $u(k)$ are the flows into or out of each reservoir.

In (4), we see that the water level behind the dam changes depending on the inflow into the reservoir and the output from the reservoir. Inflow into the reservoir includes natural inflow from tributary streams and water released from the upstream dam. Outflow from the reservoir includes water discharged through the turbine and water discharged over the spillway. We assume a tank model for the system, shown in Fig. 1, where there is a linear relationship between the volume of water in and the water level of a particular reservoir, and in which there is no time-delay between water being discharged from the upstream dam and that water arriving in the downstream reservoir. The $t_j/\Psi_j$ term in (4) is to map water flow into or out of reservoir $j$ to a proportional increase or decrease in the elevation of reservoir $j$. Since the focus of this paper is on accurately incorporating the hydropower production function, our hydraulic model is simplistic. However, incorporating a more accurate model of water flow based on a discretization of the Saint Venant open-channel flow equations [8], [9] is the subject of future research.

We introduce limits on the state and control variables corresponding to each dam $j$

$$q_{j\min} \leq q_j(k) \leq q_{j\max}$$

$$s_{j\min} \leq s_j(k) \leq s_{j\max}$$

$$w_{j\min} \leq w_j(k) \leq w_{j\max}$$

$$-\delta q_{j\min} \leq \delta q_j(k) \leq -\delta q_{j\max}$$

$$x_{j\min} \leq x_j(k) \leq x_{j\max}$$

(5) through (9)

where (5) limits turbine discharge to fixed minimum and maximum values, (6) limits spill to fixed minimum and maximum values, and (7) limits natural inflow to values determined by a forecast. Constraint (8) limits changes in turbine discharge and tailrace elevation. In the literature, researchers have chosen to minimize particular objectives [7].

III. COMPUTING HYDRAULIC HEAD

There is an observable relationship between turbine discharge and tailrace elevation. In the literature, researchers have used non-linear optimization techniques to model this relationship using polynomial functions of turbine discharge [10]–[12]. In linear modeling, however, a simple affine tailrace function has proven adequate [13].

For the system in question, we analyzed publicly available water gauge data from the United States Army Corps of Engineers recording the turbine discharge, spill, reservoir elevation, and tailrace elevation for each of the seven dams in the Mid-Columbia River hydropower system [14]. Data was available from 1997 to 2012 with hourly resolution. Empirical observations from this dataset indicate a significant linear relationship between discharge and tailrace elevation. Based on this, we model the tailrace elevation downstream of each dam $j$ using an affine function of the turbine discharge $q_j$. Since hydraulic head is the difference between the reservoir and tailrace elevation, the hydraulic head $h$ is then

$$h_j(k) = x_j(k) - (\alpha_j \cdot q_j(k - 1) + \alpha_j^0)$$

(10)

where $\alpha_j > 0$ and tailrace elevation increases linearly with turbine discharge. This relation can be reformulated as

$$h_j(0) = x_j(0) - \left( \alpha_j \cdot q_j(-1) + \alpha_j^0 \right)$$

$$h_j(k) = h_j(k - 1) + \alpha_j \cdot (q_j(k - 1) - q_j(k - 2))$$

(11a) through (11b)

where $h_j(0)$ is fixed for each optimization run according to (10) and $h_j(k)$ is calculated for $k = 1, 2, \ldots, K$. 

Fig. 1: Cross-sectional diagram of the tank model
Using the constraints from (5), we can also compute the minimum and maximum hydraulic head at each dam. The maximum hydraulic head

\[ h_j^{\text{max}} = x_j^{\text{max}} - (\alpha_j \cdot q_{j\text{min}}^{\text{max}} + \alpha_j^0) \]  

is obtained at the maximum reservoir elevation and the minimum turbine discharge (i.e., minimum tailrace elevation). The minimum hydraulic head

\[ h_j^{\text{min}} = x_j^{\text{min}} - (\alpha_j \cdot q_{j\text{max}}^{\text{min}} + \alpha_j^0) \]  

is obtained at the minimum reservoir elevation and the maximum turbine discharge (i.e., maximum tailrace elevation). The relationships (12) and (13) are important for defining the feasible set of turbine discharge \( q_j \) and hydraulic head \( h_j \) in the linearization process.

IV. HYDROPOWER PRODUCTION FUNCTION

A. Nonlinear Hydropower Production Function

In its most general form, the amount of gross electrical power extracted from a hydro turbine-generator is a non-linear function of turbine efficiency \( \eta_t \), generator efficiency \( \eta_g \), turbine discharge \( q \), and hydraulic head \( h \). This function is known as the hydropower production function (HPF). Mathematically,

\[ p(\eta_t, \eta_g, q, h) = \kappa \cdot \eta_t \cdot \eta_g \cdot q \cdot h \]  

where \( \kappa \) is a conversion constant. An example HPF is shown in Fig. 2. Generator efficiency is a non-decreasing function that varies only a couple percent over its operating range [15], whereas turbine efficiency varies substantially (10-40%) [10], [15]. For a single turbine-generator unit, the HPF is concave and known as the hill chart [10]. If turbine and generator efficiency are assumed to be functions of turbine discharge \( q \), (14) can be rewritten as

\[ p(q, h) = \kappa \cdot \eta(q) \cdot q \cdot h \]  

where \( \eta(q) \) is the combined turbine-generator efficiency.

When considering the short-term hydropower scheduling problem, the HPF (15) can be incorporated in different ways depending on the nature of the problem. Explicitly considering the efficiency term \( \eta(q) \) and the bilinear \( qh \) term generally requires mixed-integer, non-linear, or iterated linear programming that runs with a high computational cost [5], [6], [13], [16], [17]. Reducing the computational complexity entails assuming constant hydraulic head and a concave efficiency function, thus approximating the HPF with set of linear constraints at a cost of reduced accuracy [18], [19].

While ignoring short-term variations in hydraulic head or efficiency may be reasonable when scheduling hydropower with daily or hourly resolution, sub-hourly dispatch necessitates capturing dynamic effects of head and discharge. Simulating real-time hydropower dispatch for the purposes of large-scale studies also calls for a computationally efficient model. Models in the literature generally fall into one of two categories: accurate and slow, or inaccurate and fast. This warrants the development of a model and optimization technique that is simultaneously fast and accurate. The primary contribution of our work comes from the process by which we linearize the non-linear hydropower production function and integrate it into a quadratic program with linear constraints.

Efficiency curves for a given turbine are well-behaved concave functions whose best efficiency point is somewhere between the minimum and maximum output, but the exact location varies based on the design of the turbine. However, this paper does not consider the scheduling and/or dispatch of individual turbine-generator units. Thus, it is necessary to develop an efficiency curve describing multiple units aggregated together. For a multi-unit powerhouse with identical units, this is a straightforward process [10]. For a given number of online units and a given station flow, total turbine discharge is apportioned equally among the online units and the total station efficiency is calculated. This process is repeated for all combinations of online units and for all discharges between the minimum and maximum station discharge. Then, for a given station flow, the configuration that gives the highest station efficiency is selected; configurations where units are operating below their minimum discharge limit or above their maximum discharge limit are disregarded. This methodology does not account for start-up costs or minimum online times [20].

Completing this procedure yields a curve similar to the one shown in Fig. 3. The notches are the result of additional units coming online. For this example, we also observe that the efficiency curve is approximately concave from 25% to 100% of total turbine discharge when there are three or more units online. Hence, for large hydropower stations with consistently high flows and multiple units, it is not unreasonable to ignore these notches in the optimization procedure.
B. Linearizing the Hydropower Production Function

Our linearization process is almost identical to a standard segmented regression [21]. In a segmented regression, the covariates are partitioned into sections and a separate linear function is fit to each section. Since we select the number and position of each section, the problem is computationally simple. In this case, our covariates are turbine discharge \( q_i \) and hydraulic head \( h_j \), and each partition is defined as the triangle formed by a triplet of \((q_i, h_j)\) coordinates. Hence, utilizing the efficiency function for the entire hydropower plant (as shown in Fig. 3), we sample the HPF at discrete values of \( q_i \) and \( h_j \), producing a curve like the one shown in Fig. 2. Then, we create a set of \((q_j, h_j)\) breakpoints where \( h_j = h_j^{\text{max}} \) or \( h_j = h_j^{\text{min}} \), and \( q_j^{\text{min}} \leq q_j \leq q_j^{\text{max}} \). This is to ensure that our approximated HPF covers the entire feasible set of \( q_i \) and \( h_j \). We add an additional continuity constraint ensuring that the \( p_j \) computed in two adjacent sections must be equal where the two sections intersect. The optimization problem is then formulated and solved to minimize the residual sum of squares. An example of an approximated HPF is shown in Fig. 4.

C. Auxiliary Variables for Discharge and Generation

For each section of the piecewise linear approximation of the hydropower production function, we introduce auxiliary variables \( q^a_j \) and \( p^a_j \) that correspond to the turbine discharge and power produced in each section. Since no separate variables are needed for hydraulic head, there is only a single \( h_j \) for each dam. These three variables create equality constraints for each section \( i \) of the piecewise linear function of the form

\[
p^a_j(k) = \beta^{0,i}_j \cdot h_j(k) + \beta^{q,i}_j \cdot q^a_j(k)
\]

where the \( \beta \)'s are coefficients computed in the fitting process. There are \( I_j \) sections in each HPF approximation.

As discussed in the previous section, each triangular section is defined by three points of the form \((q_j, h_j, p_j)\), where each point has \( h_j = h_j^{\text{max}} \) or \( h_j = h_j^{\text{min}} \). Each section shares two common points with the section that precedes it as one moves along the \( q_j \)-axis. These two points form a line that can be projected onto the \((q_j, h_j)\) plane; the linearized HPF projected onto the \((q_j, h_j)\) plane is shown in Fig. 5 in blue. Using the relationship established in (16), the line where two adjacent sections intersect is

\[
q^i_{j, \min}(h_j) = \frac{\beta^{h,i}_j - \beta^{h,i-1}_j}{\beta^{q,i}_j - \beta^{q,i-1}_j} \cdot h_j + \frac{\beta^{0,i}_j - \beta^{0,i-1}_j}{\beta^{q,i}_j - \beta^{q,i-1}_j} - \beta^{h,i}_j.
\]

This function corresponds to the lower limit for the auxiliary variable \( q^a_j \), or

\[
q^i_{j, \min}(h_j) \leq q^a_j.
\]

Ideally, for a particular turbine discharge \( q_j \), only one auxiliary variable \( q^a_j \) will not be tight to either its upper or lower boundary. The upper limit for \( q^a_j \) is thus

\[
q^a_j \leq \begin{cases} 
q^i_{j, \min}(h_j), & \text{if } q^i_{j, \min}(h_j) < q^i_{j, \min}(h_j) \\
q^i_{j, \min}(h_j), & \text{if } q^i_{j, \min}(h_j) = q^i_{j, \min}(h_j).
\end{cases}
\]

In effect, constraint (19) maintains the ordering of the piecewise linear function. This idea is one of the standard precepts of any piecewise linearization. However, this is also the attribute that makes piecewise linear functions difficult to integrate into linear or quadratic optimization problems, because constraint (19) is obviously non-linear. We are thus forced to rewrite (19) as a linear constraint

\[
q^a_j \leq q^{i+1, \min}_j(h_j).
\]

How we adapt to this approximate constraint will be discussed in Section V.

In the case of the first section, we modify the lower limit function (17) to be the minimum turbine discharge, or

\[
q^1_{j, \min}(h_j) = q^i_{j, \min}(h_j).
\]

Similarly, in the case of the last section \( I_j \), we modify the upper limit constraint (20) to be the maximum turbine discharge, or

\[
q^a_j \leq q^{\text{max}}_j.
\]

D. Computing Total Turbine Discharge and Total Generation

The total turbine discharge is the sum of the contribution of each auxiliary turbine discharge variable. Mathematically,

\[
q_j = q^a_j + \sum_{i=2}^{I_j} \left[ q^i_j - q^{i, \min}_j(h_j) \right].
\]
Each \( q^j_i \) has a lower limit \( q^j_i \min(h_j) \), and that lower limit is subtracted from \( q^j_i \) to obtain the marginal turbine discharge from that particular section.

A similar approach is used to compute total power production. Before, we determined \( q^j_i \min(h_j) \) by projecting the linearized HPF onto the \((q_j, h_j)\) plane. Similarly, we compute \( p^j_i \min(h_j) \) by projecting the linearized HPF onto the \((p_j, h_j)\) plane. This projection is shown in Fig. 5 in red. As before, we can define the minimum power for section \( i \) of the hydropower production function as

\[
p^j_i \min(h_j) = \frac{\beta^j_q q^j_i h^i - \beta^j_h h^i}{\beta^j_q - \beta^j_q h^i} + \frac{\beta^j_h h^i - \beta^j_q h^i}{\beta^j_q - \beta^j_q h^i} \cdot h_j \tag{24}
\]

as a function of the hydraulic head \( h_j \). Then, as in (23), the total power production is

\[
p_j = p^j_i + \sum_{i=2}^{l_j} \left[ p^j_i \left( q^j_i, h^j_i \right) - p^j_i \min(h_j) \right] \tag{25}
\]

where \( p^j_i \) is calculated by (16). In other words, the total power production is the sum of the marginal power production from each section of the linearized HPF. No constraints on \( p^j_i \) are necessary since it is a linear function of \( q^j_i \) and \( h_j \), which are constrained by (5) and (10), respectively.

V. DISCUSSION OF THE OPTIMIZATION PROBLEM

The objective for our optimization problem is to minimize flow while satisfying all constraints, including power balance. This is equivalent to maximizing the efficiency of the cascade because the goal is to extract the maximum amount of energy from a unit volume of water as it moves downstream. Our optimization problem could be formulated as

\[
\min_{q^j_i, s^j_i} \sum_{k=0}^{K-1} \sum_{j=1}^{J} \lambda_j \left[ c \left( s^j_j(k) + q^j_i \max \right)^2 + \left( q^j_j(k) \right)^2 \right] \tag{26}
\]

where

\[
\lambda_j = x^j_{j\max} - \alpha^j_0 \tag{27}
\]

and \( c \) is a constant with \( c > 1 \). The weight \( \lambda_j \) is the maximum reservoir elevation minus the tailrace elevation at zero discharge. This weight is chosen to reflect the value of water behind each dam because water behind a dam with a larger hydraulic head is worth more than water behind a dam with lower hydraulic head; these weights are similar to the H/K factors sometimes used in hydropower operations. Spill \( s^j_i \) is weighted to ensure that spillage does not occur until \( q^j_i = q^j_i \max \). This is accomplished by putting \( q^j_i \max \) inside the quadratic term. The constant \( c \) is an additional weight to make certain that spill is always more costly than turbine discharge.

With all that said, minimizing turbine discharge using the formulation in (26) would be equivalent to minimizing the contribution from each auxiliary variable \( q^j_i \) (i.e., the term inside the summation in (23)). Since we approximated the upper limit constraint (19) with the less accurate constraint (20), using the objective function in (26) would result in those \( q^j_i \)'s with the steepest slopes increasing before those \( q^j_i \)'s with shallower slopes because we wish to satisfy the power balance with the least amount of turbine discharge. To avoid this behavior, we modify (26) to be

\[
\min_{q^j_i, s^j_i} \sum_{k=0}^{K-1} \sum_{j=1}^{J} \lambda_j \left[ c (s^j_j(k) + q^j_i \max)^2 + \sum_{i=1}^{l_j} (q^j_i(k))^2 \right] \tag{28}
\]

By minimizing the magnitude of each auxiliary turbine discharge variable \( q^j_i \), we can better maintain the ordering of the piecewise linear function.

Finally, we introduce a power balance constraint

\[
\sum_{j=1}^{J} p_j(k) = p_{\text{load}}(k) \quad \text{for } k = 0, 1, ..., K - 1 \tag{29}
\]

where \( p_{\text{load}}(k) \) is the net load satisfied by the cascade during a particular interval. Thus, our final optimization problem is (28) subject to MPC system constraints (4) to (9), hydraulic head constraint (11), auxiliary variable constraints (16) to (18) and (20) to (25), and the power balance constraint (29).

However, since the approximated HPF does not exactly match the actual HPF, the actual turbine discharges returned by the MPC optimization problem do not completely satisfy the power balance constraint (29). Hence, after solving the full MPC optimization problem, we run a smaller optimization in order to satisfy the power balance. The smaller optimization problem is formulated as a single time-step MPC optimization and is solved iteratively. It is formulated as

\[
\min_{q^j_i(k), s^j_i(k)} \sum_{j=1}^{J} \lambda_j \left[ (q^{(L)}_j(k) - q^{(0)}_j)^2 + (s^{(L)}_j - s^{(0)}_j)^2 \right] \tag{30}
\]

subject to

\[
x^j_j(k) = x^j_j + \frac{L_n}{\Psi_j} \left( w^j_j + q^{(L)}_j(k) - q^{(L)}_j(k) - s^{(L)}_j \right) \tag{31}
\]

\[
\sum_{j=1}^{J} \left[ q^{(L-1)}_j(k) - q^{(L-1)}_j(k) \right] = p_{\text{load}} \tag{32}
\]

\[
q^{(L)}_j \leq q^{(L)}_j \quad \text{and} \quad s^{(L)}_j \leq s^{(L)}_j \quad \text{if } \epsilon^{(L)} > 0 \tag{33}
\]

\[
q^{(L)}_j \geq q^{(L)}_j \quad \text{and} \quad s^{(L)}_j \leq s^{(L)}_j \quad \text{if } \epsilon^{(L)} < 0 \tag{34}
\]

\[
w^{(L)}_j = w^{(L)}_j \tag{35}
\]

\[
x^j_{j\min} < x^j_j < x^j_{j\max} \tag{36}
\]

for \( j = 1, ..., J \) and \( \ell = 0, ..., L - 1 \), and where

\[
\epsilon^{(L)} = \sum_{j=1}^{J} \left[ \epsilon^{(L)} q^{(L)}_j(k), h_j(k) \right] - p_{\text{load}} \tag{37}
\]

This optimization uses the variables \( q^{(0)}_j(0) \) and \( s^{(0)}_j(0) \) (for \( j = 1, 2, ..., J \)) returned by the MPC optimization to perform a single time-step MPC optimization that solves for new variables \( q^{(L)}_j(0) \) and \( s^{(L)}_j(0) \) in order to satisfy power balance and system constraints. Variables in previous iterations are denoted by the superscript \((\ell)\), and the current iteration is denoted by \((L)\). In each iteration, the power balance constraint
The basic premise behind the iterative process is that if the actual power produced for given \( q_j \) and \( s_j \) is greater than the demand (i.e., \( \epsilon > 0 \)), we put an upper limit on the turbine discharge and spill at those values. Similarly, if the actual power produced is less than the demand (i.e., \( \epsilon < 0 \)), we put a lower limit on the turbine discharge and spill at those values. These two constraints are given in (33) and (34), respectively. We can then iterate until the power balance is satisfied to within a certain tolerance. We use stopping criteria \( |\epsilon| < 2 \text{ MW} \). Inflow \( w_{ij} \) is held constant by (35), and (36) constrains the reservoir levels \( x_j^t \) at the next time-step to their minimum and maximum values.

Also, note that the system model (31) is similar to (4) except it is discretized by the simulation interval \( t_n \) instead of the optimization interval \( t_k \), where \( t_n \leq t_k \). Using a shorter simulation interval allows one to simulate a system at shorter time-steps. For example, we run our full MPC optimization every five minutes but redispach our system every 15 seconds to mimic the real-world control of hydropower generation.

VI. CASE STUDY

To demonstrate the efficacy of our model, we ran simulations using the Mid-Columbia River hydropower system as a test bed. The system runs south from the Canadian border through eastern Washington state. It consists of seven dams, including Grand Coulee Dam, the largest hydropower facility in North America. Relevant hydraulic information is given in Table I. Reservoir elevations \( x_j \) are given in feet above sea level, turbine discharges \( q \) are given in m³/s, and effective surface area \( \Psi \) is given in square kilometers. Minimum discharge was assumed to be 400 m³/s for each facility.

Hydraulic information about the Mid-Columbia system was collected from multiple private and public data sources. Surface area parameters \( \Psi_j \) were calculated from supplied forebay curves. Actual efficiency curves were not available for each Mid-Columbia facility, so we used approximated variable efficiency curves instead. As noted previously, tailrace curves were fitted to data made available by the Army Corps of Engineers [14]. In lieu of actual load data, we used load and generation data available with five-minute resolution from the Bonneville Power Administration (BPA) [22]. We simulated 15-second resolution load data by applying a cubic spline to the difference between BPA hydropower generation and wind generation in the BPA balancing area. This approximated the magnitude and volatility of the load request typically satisfied by the Mid-Columbia system. Natural inflow data was compiled from streamflow data provided by the United States Geological Survey with 15-minute resolution [23]. Natural inflows on the Mid-Columbia River are usually a small percentage of the flow on the main stem river.

Simulations were completed using two days of data from August 16-17, 2013. We used an optimization interval \( t_k \) of five minutes, simulation interval \( t_n \) of fifteen seconds, and time horizon of three hours (\( K = 36 \)). Simulations were run on a computer with an Intel Core 2 Duo 3.16 GHz and 4 GB of RAM running MATLAB 2012b (32-bit). We used the qp-minos solver called via the TOMLAB interface. The full simulation ran in 42:56 minutes, or 4.47 seconds per optimization interval. This is an order of magnitude speed-up over a similar non-linear formulation, which typically took several minutes per optimization interval to solve. The results of our simulation are shown in Fig. 6, which shows that the system was able to handle rapid ramps of system load (i.e., greater than one gigawatt per hour). Generation was, for the most part, distributed proportionally among each facility. Had we been using a time-delay in our hydraulic model, we would have expected to see non-proportional turbine discharge and power generation in anticipation of future flows. This is a focus of future research. There was no spill from any of the dams in this simulation.

Additionally, we evaluated the performance of the piecewise linear approximation of the HPF. The usable output from
the MPC model is the turbine discharge and spillage for each plant from which other variables, like tailrace elevation or generation, can be computed. However, while the piecewise linear HPF fits the true HPF with an error of no more than a couple percent, it is not an exact fit. Hence, there is some mismatch between the generation calculated using the turbine discharge from the MPC model and the actual generation needed to satisfy the system power balance. For our simulation, we computed the mismatch for the entire system at each simulation interval and plotted the histogram in Fig. 7. The left plot shows that we typically overestimate power production in the optimization process but that this mismatch is generally less than 150 MW. The right plot confirms that the mismatch is around one percent of system nameplate capacity. Considering the speed and simplicity afforded by using quadratic programming to do the optimization, we believe this is an excellent result. Nonetheless, improving the accuracy of the piecewise linearization and reducing the mismatch shown in Fig. 7 is the focus of future research.

VII. CONCLUSION

In this paper, we introduced a new method for approximating the non-linear hydropower production function and described how that approximation can be written as a set of linear constraints and integrated into a quadratic program. Using the Mid-Columbia system as a testbed, we showed that the optimization was operating as intended and that the approximated hydropower production function satisfied the system power balance constraint to within a couple percentage points of actual system load. This is an excellent result and makes us optimistic about the utility of our approach for coordinating hydropower, and we intend to continue improving and refining it. Future technical improvements to the model include the integration of a time-delay or open-channel fluid dynamics into the MPC framework. Additionally, while this research was motivated by the increased variability observed in systems with substantial wind generation, the approach focused mainly on improving sub-hourly hydro scheduling methods. Future research will use our optimization scheme to quantify the economic and technical benefits of balancing variability from renewable generation with hydropower resources, with a focus on capturing the minute-by-minute dynamics of wind, thermal, and hydropower generation.

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