Convexification of AC Optimal Power Flow

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Abstract—This overview paper summarizes the key elements of semidefinite relaxations of the optimal power flow problem, and discusses several open challenges.

I. INTRODUCTION

For our purposes an AC optimal power flow (OPF) problem, broadly defined, is a mathematical program that seeks to minimize a certain function, such as total power loss, generation cost or user disutility, subject to the Kirchhoff’s laws as well as capacity, stability and security constraints. It is fundamental as it underlies many applications in power system operations and planning. There has been a great deal of research on OPF since Carpentier’s first formulation in 1962 [1]. Extensive surveys can be found in e.g. [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13].

OPF is nonconvex and NP-hard in general. Recently a new approach to solving OPF through semidefinite relaxation is developed. To the best of our knowledge solving OPF through semidefinite relaxation is first proposed in [14] as a second-order cone program (SOCP) for radial (tree) networks and in [15] as a semidefinite program (SDP) for general networks in a bus injection model. It is first proposed in [16], [17] as an SOCP for radial networks in the branch flow model of [18], [19]. While these convex relaxations have been illustrated numerically in [14] and [15], whether or when they will turn out to be exact is first studied in [20]. Exploiting graph sparsity to simplify the SDP relaxation of OPF is first proposed in [21], [22] and analyzed in [23], [24]. See the tutorial [25], [26] for more details and pointers to the growing literature.

Solving OPF through convex relaxation offers several advantages. In particular it provides the ability to check if a solution is globally optimal. If it is not, the solution provides a lower bound on the minimum cost and hence a bound on how far any feasible solution is from optimality. Unlike approximations, if a relaxed problem is infeasible, it is a certificate that the original OPF is infeasible.

The purpose of this paper is to summarize key elements of this approach surveyed in [25], [26], and discuss some open challenges.

Notations

Let \( \mathbb{C} \) denote the set of complex numbers and \( \mathbb{R} \) the set of real numbers. For \( a \in \mathbb{C} \), \( \text{Re} a \) and \( \text{Im} a \) denote the real and imaginary parts of \( a \) respectively. For \( a, b \in \mathbb{C} \), \( a \leq b \) means \( \text{Re} a \leq \text{Re} b \) and \( \text{Im} a \leq \text{Im} b \). In general a variable without a subscript denotes a vector with appropriate components, e.g. \( s := (s_j, j=0, \ldots, n) \), \( S := (S_{jk}, (j,k) \in \mathcal{E}) \). For vectors \( x, y \), \( x \preceq y \) denotes componentwise inequality.

Matrices are usually in capital letters. The transpose of a matrix \( A \) is denoted by \( A^\top \) and its Hermitian transpose (conjugate) transpose by \( A^H \). A matrix \( A \) is Hermitian if \( A = A^H \). A is positive semidefinite (or psd), denoted by \( A \succeq 0 \), if \( A \) is Hermitian and \( x^H A x \geq 0 \) for all \( x \in \mathbb{C}^n \). For matrices \( A, B \), \( A \succeq B \) means \( A - B \) is psd. Let \( \mathbb{S}^n_+ \) be the set of all \( n \times n \) Hermitian matrices and \( \mathbb{S}^n_{++} \) the set of \( n \times n \) positive definite matrices.

A graph \( G = (N, \mathcal{E}) \) consists of a set \( N \) of nodes and a set \( \mathcal{E} \subseteq N \times N \) of edges. If \( G \) is undirected then \( (j,k) \in \mathcal{E} \) if and only if \( (k,j) \in \mathcal{E} \). If \( G \) is directed then we assume \( (j,k) \in \mathcal{E} \) only if \( (k,j) \notin \mathcal{E} \); in this case we will use \((j,k)\) and \( j \rightarrow k \) interchangeably to denote an edge pointing from \( j \) to \( k \). We sometimes use \( \tilde{G} = (\tilde{N}, \tilde{\mathcal{E}}) \) to denote a directed graph. By “\( j \sim k \)” we mean an edge \((j,k)\) if \( G \) is undirected and either \( j \rightarrow k \) or \( k \rightarrow j \) if \( G \) is directed. Sometimes we write \( j \in G \) or \((j,k) \in G \) to mean \( j \in \tilde{N} \) or \((j,k) \in \tilde{\mathcal{E}} \) respectively. A cycle \( c := (j_1, \ldots, j_k) \) is an ordered set of nodes \( j_k \in N \) so that \((j_k, j_{k+1}) \in \mathcal{E} \) for \( k = 1, \ldots, K \) with the understanding that \( j_{k+1} := j_1 \). In that case we refer to a link or a node in the cycle by \((j_k, j_{k+1}) \in c \) or \( j_k \in c \) respectively.

II. OPTIMAL POWER FLOW

In this section we describe two mathematical models of power flows. By a “mathematical model” we mean a set of variables and a set of equations relating these variables. These equations are motivated by the physical system, but mathematically, they are the starting point from which all claims are derived.

A. Bus injection model

Consider a power network modeled by a connected undirected graph \( G(N^+, \mathcal{E}) \) where \( N^+ := \{0\} \cup N \), \( N := \{1,2,\ldots,n\} \), and \( E \subseteq N^+ \times N^+ \). Each node in \( N^+ \) represents a bus and each edge in \( E \) represents a transmission
or distribution line. We use “bus” and “node” interchangeably and “line” and “edge” interchangeably. For each edge \((i, j) \in E\) let \(y_{ij} \in \mathbb{C}\) be its admittance. A bus \(j \in N^+\) can have a generator, a load, both or neither. Let \(V_j\) be the complex voltage at bus \(j \in N^+\) and \(|V_j|\) denote its magnitude. Bus 0 is the slack bus. Its voltage is fixed and we assume without loss of generality that \(V_0 = 1 \angle 0^\circ\) per unit (pu). Let \(s_j\) be the net complex power injection (generation minus load) at bus \(j \in N^+\).

The bus injection model (BIM) is defined by the following power flow equations that describe the Kirchhoff’s laws:

\[
s_j = \sum_{k \neq j} y_{jk}^H V_j (V_j^H - V_k^H), \quad j \in N^+ \tag{1}
\]

Let the set of power flow solutions \(V\) for each \(s\) be:

\[
\mathcal{V}(s) := \{V \in \mathbb{C}^{n+1} | V \text{ satisfies (1)}\}
\]

For convenience we include \(V_0\) in the vector variable \(V := (V_j, j \in N^+)\) with the understanding that \(V_0 := 1 \angle 0^\circ\) is fixed.

**Remark 1:** Bus types. Each bus \(j\) is characterized by two complex variables \(V_j\) and \(s_j\), or equivalently, four real variables. The buses are usually classified into three types, depending on which two of the four real variables are specified. For the slack bus 0, \(V_0\) is given and \(s_0\) is variable. For a generator bus (also called PV-bus), \(\text{Re}(s_j) = p_j\) and \(|V_j|\) are specified and \(\text{Im}(s_j) = q_j\) and \(\angle V_j\) are variable. For a load bus (also called PQ-bus), \(s_j\) is specified and \(V_j\) is variable.

All voltage magnitudes must satisfy:

\[
\sum_j s_j \leq |V_j|^2 \leq \tau_j, \quad j \in N^+ \tag{2}
\]

where \(\tau_j\) and \(\tau_j\) are given lower and upper bounds on the squared voltage magnitudes. Throughout this paper we assume \(\tau_j > 0\) to avoid triviality. The power injections are also constrained:

\[
\underline{s}_j \leq s_j \leq \bar{s}_j, \quad j \in N^+ \tag{3}
\]

where \(\underline{s}_j\) and \(\bar{s}_j\) are given bounds on the injections at buses \(j\).

**Remark 2:** OPF constraints. If there is no bound on the load or on the generation at bus \(j\) then \(\underline{s}_j = -\infty - i\infty\) or \(\bar{s}_j = \infty + i\infty\) respectively. On the other hand (3) also allows the case where \(s_j\) is fixed (e.g. a constant-power load), by setting \(\underline{s}_j = \bar{s}_j\) to the specified value. For the slack bus 0, unless otherwise specified, we always assume \(\underline{s}_0 = \bar{s}_0 = \tau_0 = 1\) and \(\tau_0 = -\infty - i\infty\), \(\bar{s}_0 = \infty + i\infty\). Therefore we sometimes replace \(j \in N^+\) in (2) and (3) by \(j \in N\).

We can eliminate the variables \(s_j\) from the OPF formulation by combining (1) and (3) into:

\[
\underline{s}_j \leq \sum_{k \neq j} y_{jk}^H V_j (V_j^H - V_k^H) \leq \bar{s}_j, \quad j \in N^+ \tag{4}
\]

Then OPF in the bus injection model can be defined just in terms of the complex voltage vector \(V\). Define

\[
\mathcal{V} := \{V \in \mathbb{C}^{n+1} | V \text{ satisfies (2), (4)}\} \tag{5}
\]

\(\mathcal{V}\) is the feasible set of optimal power flow problems in BIM.

Let the cost function be \(C(V)\). Typical costs include the cost of generating real power at each generator bus or line loss over the network. All these costs can be expressed as functions of \(V\). Then the problem of interest is:

\[
\min_{V} C(V) \text{ subject to } V \in \mathcal{V} \tag{6}
\]

Since (4) is quadratic, \(\mathcal{V}\) is generally a nonconvex set. OPF is thus a nonconvex problem and NP-hard to solve in general.

**B. Branch flow model**

In the branch flow model we adopt a connected directed graph \(\tilde{G} = (N^+, \tilde{E})\) where each node in \(N^+ := \{0, 1, \ldots, n\}\) represents a bus and each edge in \(\tilde{E} \subseteq N^+ \times N^+\) represents a transmission or distribution line. Fix an arbitrary orientation for \(\tilde{G}\) and let \(m := |\tilde{E}|\) be the number of directed edges in \(\tilde{G}\). Denote an edge by \((j,k)\) or \(j \rightarrow k\) if it points from node \(j\) to node \(k\). For each edge \((j,k) \in \tilde{E}\) let \(z_{jk} := 1/y_{jk}\) be the complex impedance on the line; let \(I_{jk}\) be the complex current and \(S_{jk} = P_{jk} + iQ_{jk}\) be the sending-end complex power from buses \(j\) to \(k\). For each bus \(j \in N^+\) let \(V_j\) be the complex voltage at bus \(j\). Assume without loss of generality that \(V_0 = 1 \angle 0^\circ\) pu. Let \(s_j\) be the net complex power injection at bus \(j\).

The branch flow model (BFM) in [17] is defined by the following set of power flow equations:

\[
\sum_{k \neq j} S_{jk} = \sum_{i \neq j} (S_{ij} - z_{ij} |I_{ij}|^2) + s_j, \quad j \in N^+ \tag{7a}
\]

\[
I_{jk} = y_{jk} (V_j - V_k), \quad j \rightarrow k \in \tilde{E} \tag{7b}
\]

\[
S_{jk} = V_j I_{jk}^H, \quad j \rightarrow k \in \tilde{E} \tag{7c}
\]

where (7b) is the Ohm’s law, (7c) defines branch power, and (7a) imposes power balance at each bus. The quantity \(z_{ij} |I_{ij}|^2\) represents line loss so that \(S_{ij} - z_{ij} |I_{ij}|^2\) is the receiving-end complex power at bus \(j\) from bus \(i\).

Let the set of solutions \(\tilde{x} := (S, I, V)\) of BFM for each \(s\) be:

\[
\tilde{x}(s) := \{\tilde{x} \in \mathbb{C}^{2m+n+1} | \tilde{x} \text{ satisfies (7)}\}
\]

For convenience we include \(V_0\) in the vector variable \(V := (V_j, j \in N^+)\) with the understanding that \(V_0 := 1 \angle 0^\circ\) is fixed.

Denote the variables in the branch flow model (7) by \(\tilde{x} := (S, I, V, s) \in \mathbb{C}^{2(m+n+1)}\). Define the feasible set in the branch flow model:

\[
\tilde{\mathcal{X}} := \{\tilde{x} \in \mathbb{C}^{2(m+n+1)} | \tilde{x} \text{ satisfies (7), (2), (3)}\} \tag{8}
\]
Let the cost function in the branch flow model be $C(\bar{x})$. Then the optimal power flow problem in the branch flow model is:

**OPF:**

$$\min_{\bar{x}} C(\bar{x}) \quad \text{subject to} \quad \bar{x} \in \tilde{X} \quad (9)$$

Since (7) is quadratic, $\tilde{X}$ is generally a nonconvex set. OPF is thus a nonconvex problem and NP-hard to solve in general.

**Remark 3: OPF variants.** OPF as defined in (6) and (9) is a simplified version that ignores other important constraints such as line limits, security constraints, stability constraints, and chance constraints; see extensive surveys in [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [27], [28] and a recent discussion in [29] on real-life OPF problems. Some of these can be incorporated without any change to the results in this paper (e.g. see [17], [30] for models that include shunt elements and line limits). Indeed a shunt element $y_j$ at bus $j$ can be easily included in BIM by modifying (1) into:

$$s_j = \sum_{k:j\rightarrow k} y^H_{jk} V_j (V^H_j - V^H_k) + y^H_j |V_j|^2$$

or included in BFM by modifying (7a) into:

$$\sum_{k:j\rightarrow k} S_{jk} + y^H_{jk} |V_j|^2 = \sum_{i:j\rightarrow j} (S_{ij} - z_{ij} |L_{ij}|^2) + s_j$$

**C. Equivalence**

Even though the bus injection model (1) and the branch flow model (7) are defined by different sets of equations in terms of their own variables, both are models of the Kirchhoff’s laws. It is therefore unsurprising that these two mathematical models are equivalent in the following sense. We say two sets $A$ and $B$ are equivalent, denoted by $A \equiv B$, if there is a bijection between them [31].

**Theorem 1:** $\forall(s) \equiv \tilde{X}(s)$ for any power injections $s$.

**III. Feasible sets and relaxations: BIM**

In this and the next section we derive semidefinite relaxations of OPF and clarify their relations. The cost function $C$ of OPF is usually assumed to be convex in its variables. The difficulty of OPF formulated here thus arises from the nonconvex feasible sets $\forall$ for BIM and $\tilde{X}$ for BFM. The basic approach to deriving convex relaxations of OPF is to design convex supersets of (some equivalent sets of) $\forall$ or $\tilde{X}$ and minimize the same cost function over these supersets. Different choices of convex supersets lead to different relaxations, but they all provide a lower bound to OPF. If every optimal solution of a convex relaxation happens to lie in $\forall$ or $\tilde{X}$ then it is also feasible and hence optimal for the original OPF. In this case we say the relaxation is exact.
$W_F$ is Hermitian, denoted by $W_F = W_F^H$, if $[W_F]_{jk} = [W_F]_{kj}^H$ for all $(j, k) \in F$; it is psd, denoted by $W_F \succeq 0$, if $W_F$ is Hermitian and the principal submatrices $W_F(q)$ are psd for all cliques $q$ of $F$; it is rank-$1$, denoted by rank $W_F = 1$, if the principal submatrices $W_F(q)$ are rank-1 for all cliques $q$ of $F$. We say $W_F$ is $2 \times 2$ psd (rank-1), denoted by $W_F(j,k) \succeq 0$ (rank $W_F(j,k) = 1$) if, for all edges $(j,k) \in E$, the $2 \times 2$ principal submatrices

$$[W_F](j,k) := \begin{bmatrix} [W_F]_{jj} & [W_F]_{jk} \\ [W_F]_{kj} & [W_F]_{kk} \end{bmatrix}$$

are psd (rank-1). $F$ is a chordal graph if either $F$ has no cycle or all its minimal cycles (ones without chords) are of length three. A chordal extension $c(F)$ of $F$ is a chordal graph that contains $F$, i.e., $c(F)$ has the same vertex set as $F$ but an edge set that is a superset of $F$’s edge set. In that case we call the partial matrix $W_c(F)$ a chordal extension of the partial matrix $W_F$. Every graph $F$ has a chordal extension, generally nonunique. In particular a complete supergraph of $F$ is a trivial chordal extension of $F$.

**B. Feasible sets**

We can now characterize the feasible set $\mathcal{V}$ of OPF defined in (5). Consider partial matrices $W_G$ that satisfy:

$$\sum_{k:(j,k)\in E} \gamma_k^H \left( [W_G]_{jj} - [W_G]_{jk} \right) \leq \pi_j, \quad j \in N^+$$

(11a)

$$\nu_j \leq [W_G]_{jj} \leq \pi_j, \quad j \in N^+$$

(11b)

We say that $W_G$ satisfies the cycle condition if for every cycle $c$ in $G$

$$\sum_{(j,k)\in c} \angle [W_G]_{jk} = 0 \mod 2\pi$$

(12)

When $\angle [W_G]_{jk}$ represent voltage phase differences across each line then the cycle condition imposes that they sum to zero (mod $2\pi$) around any cycle. It is proved in [31, Theorem 3] and [24], that $W_G$ has a psd rank-1 completion $W$ if and only if $W_G$ is $2 \times 2$ psd rank-1 on $G$ and satisfies the cycle condition (12), if and only if it has a chordal extension $W_c(G)$ that is psd rank-1. This leads to an exact characterization of the feasible set $\mathcal{V}$ of OPF, as follows.

Consider the following conditions on $(n+1) \times (n+1)$ matrices $W$ and partial matrices $W_{c(G)}$ and $W_G$:

$$W \succeq 0, \text{ rank } W = 1$$

(13)

$$W_{c(G)} \succeq 0, \text{ rank } W_{c(G)} = 1$$

(14)

$$W_G(j,k) \succeq 0, \text{ rank } W_G(j,k) = 1, \quad (j,k) \in E$$

(15)

Define the set of Hermitian matrices:

$$\mathcal{W} := \{ W \in \mathbb{S}^{n+1} \mid W \text{ satisfies (11),(13)} \}$$

(16)

Fix any chordal extension $c(G)$ of $G$ and define the set of Hermitian partial matrices $W_{c(G)}$:

$$\mathcal{W}_{c(G)} := \{ W_{c(G)} \mid W_{c(G)} \text{ satisfies (11),(14)} \}$$

(17)

Finally define the set of Hermitian partial matrices $W_G$:

$$\mathcal{W}_G := \{ W_G \mid W_G \text{ satisfies (11),(12),(15)} \}$$

(18)

Note that the definition of psd for partial matrices implies that $W_{c(G)}$ and $W_G$ are Hermitian. The assumption $\nu_j > 0, j \in N^+$ implies that all matrices or partial matrices have strictly positive diagonal entries.

Recall that two sets $A$ and $B$ are equivalent ($A \equiv B$) if there is a bijection between them. Even though $\mathcal{W}, \mathcal{W}_{c(G)}, \mathcal{W}_G$ are different kinds of spaces, it is proved in [31, Theorem 3] and [24], that they are all equivalent to the feasible set of OPF.

**Theorem 2:** $\mathcal{V} \equiv \mathcal{W} \equiv \mathcal{W}_{c(G)} \equiv \mathcal{W}_G$.

Theorem 2 suggests three equivalent problems to OPF.

We assume the cost function $C(V)$ in OPF depends on $V$ only through the partial matrix $W_G$. For example if the cost is total real line loss in the network then $C(V) = \sum_j \text{Re } s_j = \sum_j \sum_{(j,k)\in E} \text{Re } ([W_G]_{jj} - [W_G]_{jk})^2$. If the cost is a weighted sum of real generation power then $C(V) = \sum_j \left( c_j \text{Re } s_j + p_j^d \right)$ where $p_j^d$ are the given real power demands at buses $j$; again $C(V)$ is a function of the partial matrix $W_G$. Then Theorem 2 implies that OPF (6) is equivalent to

$$\min \limits_W C(W_G) \quad \text{s.t. } W \in \mathcal{W}_G$$

(19)

where $\mathcal{W}_G$ is any one of the sets $\mathcal{W}, \mathcal{W}_{c(G)}, \mathcal{W}_G$.

**C. Semidefinite relaxations**

Hence solving OPF (6) is equivalent to solving (19) over any of $\mathcal{W}, \mathcal{W}_{c(G)}, \mathcal{W}_G$ for an appropriate matrix variable. The difficulty with solving (19) is that the feasible sets $\mathcal{W}, \mathcal{W}_{c(G)}, \mathcal{W}_G$ are still nonconvex due to the rank-1 constraints and the cycle condition (12). Their removal leads to SDP, chordal, and SOCP relaxations of OPF respectively.

Relax $\mathcal{W}, \mathcal{W}_{c(G)}$ and $\mathcal{W}_G$ to the following convex superset:

$$\mathcal{W}^+ := \{ W \in \mathbb{S}^{n+1} \mid W \text{ satisfies (11), } W \succeq 0 \}$$

$$\mathcal{W}_{c(G)}^+ := \{ W_{c(G)} \mid W_{c(G)} \text{ satisfies (11), } W_{c(G)} \succeq 0 \}$$

$$\mathcal{W}_G^+ := \{ W_G \mid W_G \text{ satisfies (11), } W_G(j,k) \succeq 0, \quad (j,k) \in E \}$$

Define the problems:

**OPF-sdp:**

$$\min \limits_W C(W_G) \quad \text{s.t. } W \in \mathcal{W}_G^+$$

(20)

**OPF-ch:**

$$\min \limits_{W_{c(G)}} C(W_{c(G)}) \quad \text{s.t. } W_{c(G)} \in \mathcal{W}_{c(G)}^+$$

(21)

**OPF-socp:**

$$\min \limits_{W_G} C(W_G) \quad \text{s.t. } W_G \in \mathcal{W}_G^+$$

(22)
The condition \( W_G(j,k) \geq 0 \) in the definition of \( \mathbb{W}_G^+ \) is equivalent to \( |W_G|_{jk} = |W_G|_{kj}^T \) and (recall the assumption \( \nu_j > 0, j \in N^+ \))

\[
|W_G|_{jj} > 0, \ |W_G|_{kk} > 0, \ |W_G|_{jk}|W_G|_{kj} \geq |W_G|_{jk}^2
\]

This is a second-order cone and hence OPF-socp is indeed an SOCP in the rotated form.

Theorem 2 implies that if an optimal solution \( W_{\text{spd}} \) of OPF-sdp (20) is rank-1, then it is in \( \mathbb{W} \) and hence can be uniquely mapped to an optimal solution \( V_{\text{opt}} \in \mathbb{V} \) of OPF (6). Similarly if an optimal solution \( W_{\text{ch}} \) (\( W_{\text{socp}} \)) of OPF-ch (21) (OPF-socp (22)) is rank-1 (2 \( \times \) 2 rank-1 and satisfies the cycle condition (12)), then it can be uniquely mapped to an optimal solution \( V_{\text{opt}} \) in \( \mathbb{V} \) of OPF (6).

We say that a set \( A \) is an effective subset of a set \( B \), denoted by \( A \sqsubseteq B \), if, given a (partial) matrix \( a \in A \), there is a (partial) matrix \( b \in B \) that has the same cost \( C(a) = C(b) \). We say \( A \) is similar to \( B \), denoted by \( A \equiv B \), if \( A \sqsubseteq B \) and \( B \sqsubseteq A \). Note that \( A \equiv B \) implies \( A \equiv B \) but the converse may not be true.

The feasible set of OPF (6) is an effective subset of the feasible sets of the relaxations; moreover these relaxations have similar feasible sets when the network is radial. This is a slightly different formulation of the same results in [31, 24].

**Theorem 3:** \( \mathbb{V} \sqsubseteq \mathbb{W} \sqsubseteq \mathbb{W}_{c(G)}' \sqsubseteq \mathbb{W}_{c(G)}^+ \sqsubseteq \mathbb{W}^+ \). If \( G \) is a tree then \( \mathbb{V} \sqsubseteq \mathbb{W} \sqsubseteq \mathbb{W}_{c(G)}' \approx \mathbb{W}_{c(G)}^+ \approx \mathbb{W}_G^+ \).

The various feasible sets and their equivalence are illustrated in Figure 1.

Fig. 1: Equivalent feasible sets \( \mathbb{V}, \mathbb{W}, \mathbb{W}_{c(G)}, \mathbb{W}_G, X \) and their semidefinite relaxations \( \mathbb{W}^+, \mathbb{W}_{c(G)}', \mathbb{W}_{c(G)}^+ \) and \( X^+ \) (The sets \( X \) and \( X^+ \) are defined in Section IV). For radial networks these relaxations are similar.

Let \( C_{\text{opt}}, C_{\text{spd}}, C_{\text{ch}}, C_{\text{socp}} \) be the optimal values of OPF (6), OPF-sdp (20), OPF-ch (21), OPF-socp (22) respectively. Theorem 2 and Theorem 3 directly imply

**Corollary 4:** \( C_{\text{opt}} \geq C_{\text{spd}} \geq C_{\text{ch}} \geq C_{\text{socp}} \). If \( G \) is a tree then \( C_{\text{opt}} \geq C_{\text{spd}} \geq C_{\text{ch}} = C_{\text{socp}} \).

**Remark 4:** Theorem 3 and Corollary 4 imply that for radial networks one should always solve OPF-socp since it is the tightest and the simplest relaxation of the three. For mesh networks there is a tradeoff between OPF-socp and OPF-ch/OPF-sdp: the latter is tighter but requires heavier computation. Between OPF-ch and OPF-sdp, OPF-ch is usually preferable as they are equally tight but OPF-ch is usually much faster to solve for large sparse networks. See [21], [22], [24], [23], [35] for numerical studies that compare these relaxations.

**D. Solution recovery**

When the convex relaxations OPF-sdp, OPF-ch, OPF-socp are exact, i.e., if their optimal solutions \( W_{\text{spd}}, W_{\text{ch}}, W_{\text{socp}} \) happen to lie in \( \mathbb{W}, \mathbb{W}_{c(G)}, \mathbb{W}_G \) respectively, then an optimal solution \( V_{\text{opt}} \) of the original OPF can be recovered from these solutions. Indeed the recovery method works not just for an optimal solution, but any feasible solution that lies in \( \mathbb{W}, \mathbb{W}_{c(G)}, \mathbb{W}_G \). Moreover, given a \( W \in \mathbb{W} \) or a \( W_{c(G)} \in \mathbb{W}_{c(G)} \), the construction of \( V \) depends on \( W \) or \( W_{c(G)} \) only through their submatrix \( W_G \). We hence describe the method for recovering the unique \( V \) (unique since \( G \) is connected and \( V_0 \) is fixed) from a \( W_G \), which may be a partial matrix in \( \mathbb{W}_G \) or the submatrix of a (partial) matrix in \( \mathbb{W} \) or \( \mathbb{W}_{c(G)} \).

Let \( T \) be an arbitrary spanning tree of \( G \) rooted at bus 0. Let \( P_j \) denote the unique path from node 0 to node \( j \) in \( T \). Recall that \( V_0 = 1 \angle 0^\circ \) without loss of generality. For \( j = 1, \ldots, n \), let

\[
|V_j| := \sqrt{|W_G|_{jj}} \quad \angle V_j := - \sum_{(i,k) \in P_j} \angle |W_G|_{ik}
\]

Then it can be checked that \( V \) is in the feasible set \( \mathbb{V} \) of OPF defined in (5).

**IV. FEASIBLE SETS AND RELAXATIONS: BFM**

We now present an SOCP relaxation of OPF (9) in BFM proposed in [16], [17].

**A. Feasible sets**

Consider the following set of equations in the variables \( x := (S, \ell, v, s) \) in \( \mathbb{R}^{3|m+n+1} \):

\[
\begin{align*}
\sum_{j \leftarrow k} S_jk &= \sum_{i \rightarrow j} (S_{ij} - z_{ij} \ell_{ij}) + s_j, \quad j \in N^+ \quad (23a) \\
v_j - v_k &= 2 \text{Re}(z_{jk}^H S_{jk}) - |z_{jk}|^2 \ell_{jk}, \quad j \rightarrow k \in \tilde{E} \quad (23b) \\
v_j \ell_{jk} &= |S_{jk}|^2, \quad j \rightarrow k \in \tilde{E} \quad (23c)
\end{align*}
\]

and define the solution set as:

\[
\mathcal{X}_{nf} := \{x \in \mathbb{R}^{3|m+n+1} | x \text{ satisfies (2), (3), (23)}\}
\]

Note that the vector \( v \) includes \( v_0 \) and \( s \) includes \( s_0 \). The model (23) is first proposed in [18], [19]. It is a relaxation of BFM (7) in the sense that the feasible set \( \mathcal{X} \) of OPF
in (9) is an effective subset of \( \mathcal{X}_{nc} \), \( \tilde{\mathcal{X}} \subseteq \mathcal{X}_{nc} \). We now characterize the subset of \( \mathcal{X}_{nc} \) that is equivalent to \( \tilde{\mathcal{X}} \).

Given an \( x := (S, \ell, v, s) \in \mathbb{R}^{3(m+n+1)} \) define \( \beta(x) \in \mathbb{R}^m \) by

\[
\beta_{jk}(x) := \angle(v_j - x_j^H S_{jk}, j \rightarrow k) \in \tilde{E} \quad (24)
\]

Even though \( x \) does not include phase angles of \( V \), \( x \) implies a phase difference across each line \( j \rightarrow k \in \tilde{E} \) given by \( \beta_{jk}(x) \). The subset of \( \mathcal{X}_{nc} \) that is equivalent to \( \tilde{\mathcal{X}} \) are those \( x \) for which there exists \( \theta \) such that \( \theta_j - \theta_k = \beta_{jk}(x) \).

To state this precisely let \( B \) be the \( m \times n \) (transposed) reduced incidence matrix of \( \tilde{G} \):

\[
B_{lj} = \begin{cases} 
1 & \text{if edge } l \in \tilde{E} \text{ leaves node } j \\
-1 & \text{if edge } l \in \tilde{E} \text{ enters node } j \\
0 & \text{otherwise}
\end{cases}
\]

where \( j \in N \). Consider the set of \( x \in \mathcal{X}_{nc} \) such that

\[
\exists \theta \text{ that solves } B \theta = \beta(x) \mod 2\pi \quad (25)
\]
i.e., \( \beta(x) \) is in the range space of \( B \mod 2\pi \). A solution \( \theta(x) \), if exists, is unique in \((-\pi, \pi]^n\). This is the cycle condition in BFM: it means that the angle differences implied by \( x \) must sum to zero around every cycle. Moreover it is proved in [17, Theorems 2 and 4] that the subset of \( \mathcal{X}_{nc} \) that satisfies the cycle condition (25) is equivalent to the feasible set \( \tilde{\mathcal{X}} \) of OPF (9) in BFM. Specifically define the set

\[
\mathcal{X} := \{ x \in \mathbb{R}^{3(m+n+1)} \mid x \text{ satisfies (2), (3), (23), (25)} \}
\]

Then \( \tilde{\mathcal{X}} \equiv \mathcal{X} \subseteq \mathcal{X}_{nc} \).

The set \( \mathcal{X}_{nc} \) is nonconvex because of the quadratic equalities in (23c). Relax them to inequalities:

\[
v_j \ell_{jk} \geq |S_{jk}|^2, \quad (j,k) \in \tilde{E} \quad (26)
\]

and define the set:

\[
\mathcal{X}^+ := \{ x \in \mathbb{R}^{3(m+n+1)} \mid x \text{ satisfies (2), (3), (23a), (23b), (26)} \}
\]

Note that \( \mathcal{X}^+ \) is a second-order cone in the rotated form.

Throughout this paper we assume the cost function \( C(\tilde{x}) \) in OPF (9) depends on \( \tilde{x} \) only through \( x := h(\tilde{x}) \). For example for total real line loss \( C(\tilde{x}) = \sum_{i(k,j) \in \tilde{E}} Re c_{ij} \ell_{jk} \).

If the cost is a weighted sum of real generation power then \( C(\tilde{x}) = \sum_{i} (c_i p_{ij} + p_{ij}^0) \) where \( p_{ij} \) are the real parts of \( s_i \) and \( p_{ij}^0 \) are the given real power demands at buses \( j \); again \( C(\tilde{x}) \) depends only on \( x \). The three sets \( \mathcal{X}, \mathcal{X}_{nc}, \mathcal{X}^+ \) define the following problems:

**OPF:**

\[
\min_x C(x) \quad \text{subject to } \quad x \in \mathcal{X} \quad (27)
\]

**OPF-nc:**

\[
\min_x C(x) \quad \text{subject to } \quad x \in \mathcal{X}_{nc} \quad (28)
\]

**OPF-socp:**

\[
\min_x C(x) \quad \text{subject to } \quad x \in \mathcal{X}^+ \quad (29)
\]

The next theorem follows from the results in [17] and implies that OPF (9) is equivalent to minimization over \( \mathcal{X} \) and OPF-socp is its SOCP relaxation. Moreover for radial networks voltage and current angles can be ignored and OPF (9) is equivalent to OPF-nc.

**Theorem 5:** \( \tilde{\mathcal{X}} \equiv \mathcal{X} \subseteq \mathcal{X}_{nc} \subseteq \mathcal{X}^+ \). If \( \tilde{G} \) is a tree then \( \tilde{\mathcal{X}} = \mathcal{X} = \mathcal{X}_{nc} \subseteq \mathcal{X}^+ \).

Let \( C_{opt} \) be the optimal cost of OPF (9) in the branch flow model. Let \( C_{opt}, C_{nc}, C_{socp} \) be the optimal costs of OPF (27), OPF-nc (28), OPF-socp (29) respectively defined above. Theorem 5 implies

**Corollary 6:** \( C_{opt} \geq C_{nc} \geq C_{socp} \). If \( \tilde{G} \) is a tree then \( C_{opt} = C_{nc} \geq C_{socp} \).

### B. Equivalence

Theorem 5 establishes a bijection between \( \mathcal{X} \) and the feasible set \( \tilde{\mathcal{X}} \) of OPF (9) in BFM. Theorem 2 establishes a bijection between \( \mathcal{W}_G \) and the feasible set \( \mathcal{V} \) of OPF (6) in BIM. Theorem 1 hence implies that \( \mathcal{X} \equiv \tilde{\mathcal{X}} \equiv \mathcal{V} \equiv \mathcal{W}_G \).

Moreover their SOCP relaxations are equivalent in these two models [31], [24]. Hence SOCP is exact in BFM if and only if it is exact in BFM; see Figure 1.

### V. Exact relaxations

We say a semidefinite relaxation is exact if an optimal solution of the original nonconvex OPF can be recovered from every optimal solution of the relaxation (see a more precise definition in [26]). Network topology turns out to play a critical role in determining whether a relaxation is exact. For radial networks, there are roughly three types of sufficient conditions in the literature that guarantee the exactness of semidefinite relaxations. These conditions are generally not necessary and they have implications on allowable power injections, voltage magnitudes, or voltage angles:

- **A. Power injections:** These conditions require that not both constraints on real and reactive power injections be binding at both ends of a line.
- **B. Voltages magnitudes:** These conditions require that the upper bounds on voltage magnitudes not be binding. They can be enforced through affine constraints on power injections.
- **C. Voltage angles:** These conditions require that the voltage angles across each line be sufficiently close. This is needed also for stability reasons.

These conditions and their references are summarized in Tables I for radial networks and II for mesh networks. See [26] for details. Some of these sufficient conditions are proved using BIM and others using BFM. Since these two models are equivalent (Theorem 1), these sufficient conditions apply to both models. Moreover it is proved
TABLE I: Sufficient conditions for radial (tree) networks.

<table>
<thead>
<tr>
<th>type</th>
<th>condition</th>
<th>model</th>
<th>reference</th>
<th>remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>power injections</td>
<td>BIM, BFM</td>
<td>[36], [30], [37], [38], [39]</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>voltage magnitudes</td>
<td>BFM</td>
<td>[41], [42], [43], [44]</td>
<td>allows general injection region</td>
</tr>
<tr>
<td>C</td>
<td>voltage angles</td>
<td>BIM</td>
<td>[45], [46]</td>
<td>makes use of branch power flows</td>
</tr>
</tbody>
</table>

TABLE II: Sufficient conditions for mesh networks

<table>
<thead>
<tr>
<th>network</th>
<th>condition</th>
<th>reference</th>
<th>remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>with phase shifters</td>
<td>type A, B, C</td>
<td>[17, Part II], [47]</td>
<td>equivalent to radial networks</td>
</tr>
<tr>
<td>direct current</td>
<td>type A</td>
<td>[17, Part I], [20], [48]</td>
<td>assumes nonnegative voltages</td>
</tr>
<tr>
<td></td>
<td>type B</td>
<td>[49], [50]</td>
<td>assumes nonnegative voltages</td>
</tr>
</tbody>
</table>

in [44] using BFM that when the cost function is convex then exactness of the SOCP relaxation implies uniqueness of the optimal solution for radial networks. Hence any of the three types of sufficient conditions guarantees that, for a radial network with a convex cost function, there is a unique optimal solution and it can be computed by solving an SOCP. Empirical evidences suggest some of these conditions are likely satisfied in practice. This is important as most power distribution systems are radial.

These conditions are insufficient for general mesh networks because they cannot guarantee that an optimal solution of a relaxation satisfies the cycle conditions. It is shown in [17, Part II], however, that these conditions are sufficient for mesh networks that have tunable phase shifters at strategic locations.

VI. EXTENSIONS AND CHALLENGES

In this section we mention four areas to which convex relaxation is being applied or should be extended.

A. Higher-order SDP

As alluded to above, no sufficient conditions are yet known for exact relaxation of general mesh networks without tunable phase shifters. Indeed counterexamples are discussed in, e.g., [70], [71], [72], where SDP relaxation is not exact. This has motivated the application of higher-order semidefinite relaxations on the Lasserre hierarchy for polynomial optimization to OPF [73], [74], [75], [76]. By going up the hierarchy, the relaxations become tighter and their solutions approach a global optimal of the original polynomial optimization [77], [78]. This however comes at the cost of significantly higher runtime. Techniques are proposed in [75], [76] to reduce the problem sizes, e.g., by exploiting sparsity or adding redundant constraints [79], [80], [76] or applying higher-order relaxations only on (typically small) subnetworks where constraints are violated [75].

B. Stochastic OPF

As renewable generation such as wind and solar power continues to grow, volatility and uncertainty will increase in the future. Stochastic OPF introduces an additional source of nonconvexity that can be dealt with through convex relaxation. There is a large literature on this topic, including some recent works on, among others, chance constrained OPF, e.g., [51], [52] as well as stochastic control over multiple periods, e.g., [53], [54].

C. Distributed OPF

We envision a future grid that has hundreds of millions of distributed energy resources such as solar panels, wind turbines, inverters, electric vehicles, smart buildings and appliances, and storage devices. Unlike most endpoints today that are passive loads, these are active endpoints that can generate, sense, compute, communicate and actuate. Such a large network of active endpoints must, and can, be stabilized and optimized to improve reliability, robustness, efficiency, and security. This is difficult because the control points are distributed and large in number, and uncertainty calls for real-time control. This suggests distributed feedback control that relies on local measurements in real time, local communications, and local decisions. Distributed solution of OPF will be part of this control architecture, e.g. [55], [56], [57], [58], [59], [60].

Distributed control based on local sensing and communication is not only important for scalability, it can also help with another common problem: network parameters needed for centralized solutions are often unavailable or inaccurate. One approach to deal with this data problem is to design algorithms where decisions are based on partial state information that can be locally measured or communicated from neighbors. The challenge is to design local algorithms with a global perspective such that these local decisions collectively optimize a global objective. See, e.g., [61] for an illustration of this approach (in a different context) where intelligent loads participate in primary frequency regulation by adjusting their consumption.

These conditions are insufficient for general mesh networks because they cannot guarantee that an optimal solution of a relaxation satisfies the cycle conditions. It is shown in [17, Part II], however, that these conditions are sufficient for mesh networks that have tunable phase shifters at strategic locations.
dynamically based only on local frequency deviations. In that case, a load does not need any information about the network, such as topology, current state of generators or loads, but only its local frequency deviation which contains exactly the right information about global power imbalance for it to make a local decision that turns out to be socially optimal.

D. Multiphase OPF

The results we have discussed so far are all for single-phase networks. Distribution systems, however, are typically radial, multiphase and unbalanced. Many power flow models and solution methods have been proposed in the literature for multiphase unbalanced networks, e.g., [62], [63], [64], [65]. To simplify notation, we assume without loss of generality that all buses and all lines have three phases.

Models for a three-phase network in the literature mostly correspond to the generalization of the bus injection model discussed in this paper from single phase to multiple phases, as follows. Consider a 3-phase radial network with a set \( N^+ := \{0, 1, \ldots, n\} \) of \( n+1 \) buses and a set \( E \) of \( n \) lines. Each 3-phase line \((j,k)\) is modeled by a phase admittance matrix [66]

\[
y_{jk} := \begin{bmatrix}
y_{ja} & y_{jb} & y_{jc} \\
y_{ka} & y_{kb} & y_{kc} \\
y_{na} & y_{nb} & y_{nc}
\end{bmatrix}
\]

where \( y_{jk} \) is symmetric and \( y_{kj} = y_{jk} \). The bus injection model of a 3-phase network consists of two modeling assumptions. First, for each line \((j,k) \in E\), the current vector \( I_{jk} \in \mathbb{C}^3 \) and the voltage vectors \( V_j, V_k \in \mathbb{C}^3 \) on both ends of the line satisfy the Ohm’s law [66]:

\[
I_{jk} = y_{jk}(V_j - V_k), \quad (j,k) \in E \quad (30a)
\]

The injection current \( I_j \in \mathbb{C}^3 \) is defined to be \( I_j := \sum_{k,j} I_{jk} \) and hence the current injection at bus \( j \) phase \( \phi \) is:

\[
I_j^\phi = \sum_{k,j} \sum_{\phi'} y_{jk}^\phi (V_j^{\phi'} - V_k^{\phi'})
\]

\[
= \left( \sum_{k,j} y_{jk}^\phi \right) V_j^\phi + \sum_{k,j} \sum_{\phi' \neq \phi} y_{jk}^{\phi'} V_j^{\phi'} - \sum_{k,j} \sum_{\phi' \neq \phi} y_{jk}^{\phi'} V_k^{\phi'} \quad (30b)
\]

Second, for each bus \( j \in N^+ \) and each phase \( \phi \in \{a,b,c\} \), power balance is modeled by:

\[
s_j^\phi = V_j^\phi (I_j^\phi)^H \quad (30c)
\]

These modeling assumptions (30) yield the bus injection model for a 3-phase network:

\[
s_j^\phi = \sum_{k,j} \sum_{\phi'} (y_{jk}^{\phi'})^H V_j^\phi (V_j^{\phi'} - V_k^{\phi'})^H + \\
\sum_{k,j} \sum_{\phi', \phi'' \neq \phi} (y_{jk}^{\phi'})^H V_j^{\phi'} (V_k^{\phi''} - V_j^{\phi''})^H
\]

\[
j \in N^+, \phi \in \{a,b,c\} \quad (31)
\]

which is the same as BIM (1), if we identify each \((j,\phi)\) as a separate node.

Indeed, it has already been observed in [62], [63] that a 3-phase network has an equivalent single-phase circuit model that can be represented by an undirected graph \( \overline{G} := (\overline{N}^+, \overline{E}) \) with \( 3(n+1) \) nodes and \( 12n+3 \) links. For each bus \( j \in N^+ \), there are 3 node in \( N^+ \) indexed by \((j,\phi)\) for \( \phi \in \{a,b,c\} \). For each line \((j,k) \in E\), there are 9 links in \( \overline{E} \) indexed by \((j,\phi, k, \phi')\) for \( \phi, \phi' \in \{a,b,c\} \) and 6 links in \( \overline{E} \) indexed by \((j,\phi, k, \phi')\) and \((k,\phi, k, \phi')\) with \( \phi \neq \phi' \); see Figure 2. Since the 3-phase network is radial, the single-phase equivalent model has a radial macro-structure in which each line \((j,k) \in E\) is represented as a clique (complete subgraph) in \( \overline{G} \). Hence the network graph \( \overline{G} \) of the single-phase equivalent model is chordal.

All the results on BIM discussed in Section III are thus applicable to the single-phase equivalent model (31). Indeed SDP relaxation is proposed in [67] for solving OPF for general multiphase networks and a distributed algorithm based on ADMM is developed. Since \( \overline{G} \) is a chordal graph, chordal relaxation is proposed in [68], [69] for solving OPF for multiphase radial networks. This is much more efficient for large networks as it solves for a partial matrix instead of a full matrix, as follows.

Define the Hermitian partial matrix \( W_{\overline{G}} \) by:

\[
W_{\overline{G}} := \left( \begin{array}{cc} W_{\overline{G}}^{a\phi} & (j,\phi) \in \overline{N}^+ \quad, W_{\overline{G}}^{b\phi'} \quad, (j,\phi, k, \phi') \in \overline{E} \end{array} \right)
\]

Motivated by (31), consider the set of partial matrices \( W_{\overline{G}} \)
that satisfy
\[
{s^\phi_j} = \sum_{k,k \sim j, \phi \neq \phi'} \left( y_{jk}^\phi \right)^H \left( W_{G_{kk}}^\phi \phi - [W_{G_{kk}}^\phi \phi'] \right) + \\
\sum_{k,k \sim j, \phi \neq \phi'} \left( -y_{jk}^\phi \right)^H \left( W_{G_{kk}}^\phi \phi - [W_{G_{kk}}^\phi \phi'] \right)
\]
\[j \in N^+, \phi \in \{a, b, c\}\]

(32a)

Consider the constraints
\[
\begin{align*}
\sum_{j \in N^+} s^\phi_j & \leq s^\phi, \quad j \in N^+ \\
\sum_{j \in N^+} W^\phi_{G_{jj}} & \leq \gamma^\phi, \quad j \in N^+
\end{align*}
\]

(32b)

(32c)

Then the chordal relaxation OPF-ch (21) applied to the single-phase equivalent of a 3-phase network takes the form
\[
\min_{W_G} C(W_G) \text{ s.t. } W_G \in W^+_G
\]

where the feasible set is
\[
W^+_G := \{ W_G | W_G \text{ satisfies (32), } W_G \succeq 0\}
\]

The set of maximal cliques of \(G\) correspond to the subgraphs of \(G\) associated with each line \((j,k) \in E\). Let \(W_G(j,k)\) denote the \(6 \times 6\) principal submatrix of the partial matrix \(W_G\) corresponding to the maximal clique associated with line \((j,k) \in E\) (see Figure 2):

\[
W_G(j,k) := \begin{bmatrix} W_{G_{jj}} & W_{G_{jk}} \\ W_{G_{kj}} & W_{G_{kk}} \end{bmatrix}
\]

where \([W_{G_{jj}}], [W_{G_{kk}}], [W_{G_{jk}}], [W_{G_{kj}}]\) are \(3 \times 3\) principal submatrices of \(W_G\) (as opposed to being complex scalars in the single-phase case; see Section III-A). Recall that \(W_G \succeq 0\) means that these principal submatrices \(W_G(j,k)\) are psd for all lines \((j,k) \in E\). Theorem 2 then implies that if an optimal solution \(W_G\) is also rank-1 (i.e., the principal submatrices \(W_G(j,k)\) are rank-1 for all lines \((j,k) \in E\)), then an optimal solution of OPF can be recovered. In that case, for each 3-phase line \((j,k) \in E\), there are 3-phase voltages \(V_j, V_k \in \mathbb{C}^3\) such that the principal submatrix \(W_G(j,k)\) satisfies:

\[
W_G(j,k) = \begin{bmatrix} V_j \\ V_k \end{bmatrix} \begin{bmatrix} V_j^H \\ V_k^H \end{bmatrix}
\]

Moreover, given a rank-1 partial matrix \(W_G\) that solves the chordal relaxation of OPF, the algorithm in Section III-D can be applied to recover the unique full voltage vector \(V := \begin{bmatrix} V_j \\ V_k \end{bmatrix} \in \mathbb{C}^N, \phi \in \{a, b, c\}\) that is optimal for the original multiphase OPF.\(^2\) Clearly the partial matrix \(W_G\) is a submatrix of the rank-1 matrix \(VV^H\).

No sufficient condition is yet known that guarantees the exactness of SDP relaxation for a 3-phase radial network. The sufficient conditions listed in Table I that guarantee the exactness of semidefinite relaxations, however, apply only to (single-phase) radial networks, but the single-phase equivalent model of a 3-phase radial network is mesh. Empirically, however, the SDP/chordal relaxation of OPF for multiphase networks often yield globally optimal solutions.

Importantly, BIM is ill-conditioned. From our experience with IEEE test systems, the problem of numerical instability with BIM is much worse for the multiphase unbalanced case than it is for the single-phase case. In [69], BFM is extended from single phase to multiple phases and a semidefinite relaxation is proposed for radial networks. Not only does the relaxation yield globally optimal solutions in case studies, BFM is also much more stable numerically on all these cases than the chordal relaxation based on BIM. See [69] also for a linear approximation of the multiphase BFM that generalizes the Simple DistFlow equations of [19] from a single phase to multiple phases.

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